STUDIES OF HEAT TRANSFER IN THE FIRST AND THE SECOND ORDER BOUNDARY LAYER FLOWS

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DEPARTMENT OF AERONAUTICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

1969

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A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

BY

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to the

DEPARTMENT OF AERONAUTICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

1969

CERTIFICATE

This is to certify that the present work STUDIES OF HEAT TRANSFER IN THE FIRST AND THE SECOND ORDER BOUNDARY LAYER FLOWS has been carried out under my supervision and has not been submitted elsewhere for the award of a degree or diploma.

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ACKNOWLEDGEMENT

It is a great pleasure to thank Dr. Madan Mohan Oberai for his invaluable guidance, comments, criticisms and encouragement throughout the course of present work. I also appreciate his liberal attitude towards the freedom of thought which I enjoyed during his guidance.

I also wish to thank Dr. Roddam Narasimha for introducing me to this subject and for many stimulating discussions during his brief visit to Kanpur. I also take this opportunity to thank all the members of 'Aerodynamic Group' particularly Dr. A.C.Jain for many interesting discussions.

Further, I acknowledge with gratitude the continuous interest and encouragement of Dr. P.N. Murthy and Dr. S.M. Ramachandra during the course of this work.

I am also grateful to Mrs. Rachel Oberai for her generous help. Finally, thanks are due to Mr. J.U.Siddiqui, Mr. M.Z. Siddiqui and Mr. B.K. Gupta for their interest in the work and due to Mr. J. K. Misra of Graphic Arts for his cooperation in bringing out the work in the present form.

Noor Afzal

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NOMENCLATURE

a, b, w,	indices of power laws defined by equation (3.7)
$B(\eta)$,	Blasius solution for flat plate
$B_1(\Psi_1)$,	Bernoulli function defined by equation (2.11a)
C,	compressibility factor
C_f,	skin friction coefficient
C_p,	specific heat at constant pressure
d,	a constant defined as $= b(1 + m)$
$D(\beta,\sigma)$,	heat transfer rate at the wall defined by the equation (4.26a)
e,	a constant defined as $= -(a+w)(1+m)$
E,	Eckert number defined as $= U_c^2/h_c$
f',	dimensionless velocity in s direction due to first order boundary layer
F',	correction to f due to second order boundary layer
${}_2F_1$,	hypergeometric function defined as ${}_2F_1(a,b,c,x) = \sum_{n=0} (a)_n(b)_n x^n/\left[(c)_n\ n!\right]$ where $(a)_n = \Gamma(a+n)/\Gamma(a)$
g,	dimensionless first order boundary layer enthalpy
G,	change in g due to second order boundary layer
h,	enthalpy of the fluid
$h_1, h_2, ..$	first and second order enthalpy in the inner (boundary layer) flow defined in (2.17)

$H_1, H_2, \ldots$	first and second order enthalpy in the outer flow defined in (2.7)
$J(\eta)$,	a function defined as $= f''^2 + \beta f' (1-f'^2)$.
j,	a number equal to zero for two dimensional flow and unity for axisymmetric flow
k,	thermal conductivity of the fluid
k_1,	longitudinal curvature parameter defined as $= K \sqrt{(2\xi)}/(U_1 r^j)$
k_t,	transverse curvature parameter defined as $= (j \cos\theta/r) \sqrt{(2\xi)}/(U_1 r^j)$
K,	longitudinal surface curvature of the body
L,	characteristic length of the body
L_r,	operator defined as $= u_r\, \partial/\partial s + v_r\, \partial/\partial N$, $r = 1, 2, \ldots$
m,	a constant defined as $= (b-a-w)/(1-b+a+w)$
M_r, M_{rr},	functions defined by equations (4.23) and (4.34)
n,	coordinate normal to body
$n_\alpha(\beta)$,	a function defined as$=N_\alpha(\beta, \infty)$
N,	boundary layer (inner) variable defined as $=R^{1/2}\, n$
$N(\alpha, x)$,	related gamma function defined as $= \alpha \int_0^1 e^{-xt} (1-t)^{\alpha-1}\, dt$
$N_m(\alpha, x)$,	function defined as $= \int_0^x t^m N(\alpha, t)\, dt$
P,	pressure of fluid
$P_1, P_2, \ldots$,	first and second order pressures in the outer flow defined in (2.7)
$p_1, p_2, \ldots$,	first and second order pressures in the inner flow defined in (2.17)

q,	the heat transfer rate at the wall
$q_1, q_2, \ldots$,	first and second order heat transfer at the wall defined by equation (5.4)
$Q(\ )$,	differential operator defined as $= d^2/dy^2 + (ay^2/2)\, d/dy$
$Q_1(\ )$,	differential operator defined as $= d^2/d\bar{z}^2 - (\beta\bar{z}^3/3)\, d/d\bar{z}$
r,	radius of body in axisymmetric flow
r_f,	the recovery factor
R,	characteristic Reynolds number of the flow defined as $(\rho UL/\mu)_c$
s,	coordinate along the body
$S_1(\Psi_1)$,	Stagnation enthalpy function
$\underset{\sim}{U}$,	velocity vector with components u, v
u	velocity component in s direction
$u_1, u_2, \ldots$,	first and second order velocities in s direction in inner flow defined in (2.7)
$\underset{\sim}{U}_1, \underset{\sim}{U}_2, \ldots$,	first and second order velocities (in vector notation) in the outer flow defined in (2.17)
$U_1(s,0)$,	velocity of the slip at the wall from first order outer flow
$U_2(s,0)$,	velosity of slip at the wall from the second order outer flow
v,	velocity component in n-direction
$v_1, v_2, \ldots$,	first and second order velocity in n-direction in the inner flow defined in (2.17)

$V_2(s,\ 0)$, second order normal velocity at the wall defined in (2.29)

W_1, W_h, W_o, functions proportional to heat transfer due to wall temperature in low, high and order unity values of Prandtl number defined by equations (3.42),(3.58) and (3.72) respectively.

$\underset{\sim}{X}$, distance vector

$Y_t(\quad)$, a differential operator defined as

$$= \frac{d^3}{d\eta^3} + f_o \frac{d^2}{d\eta^2} - 2(\beta_o + t)\ f_o' \frac{d}{d\eta} + (1+2t)\ f_o'',\ t=0,1,..$$

y, $\bar{z}$, inner variables in high Prandtl number flows

$Z_t(\quad)$, a differential operator defined as

$$= \sigma^{-1} \frac{d^2}{d\eta^2} + f_o \frac{d}{d\eta} - 2tf_o' \quad , \quad t = 0,1,2,..$$

z, outer variable in low Prandtl number flows defined as $= \zeta - \sigma_\infty^{-1/2}\ \alpha$.

Greek Symbols:

α, proportional to displacement thickness defined by $\lim\limits_{\eta\to\infty} (\eta - f)$

α_o, α_1, defined as $\lim\limits_{\eta\to\infty} (\eta - f_o)$ and $\lim\limits_{\eta\to\infty} (-f_1)$

β, principal velocity function defined as

$$= \frac{2}{U_1^{\,2} r^{2j}} \frac{dU_1}{ds} \int_0^s U_1 r^{2j}\ ds, \text{ with } U_1 = U_1(s,\ 0)$$

$\beta_o, \beta_1, ..$, coefficients in expansion of principal function defined by (5.21)

$\gamma(\alpha, x)$, the incomplete gamma function defined by

$$= \int_0^x e^{-t}\, t^{\alpha-1}\, dt$$

$\gamma_1(\alpha, x)$, the incomplete gamma function with one negative argument defined as

$$= \int_0^x e^{t}\, t^{\alpha-1}\, dt$$

$\Gamma(\alpha)$, gamma function defined as $= \gamma(\alpha, \infty)$

δ, displacement thickness

$\delta_1, \delta_2 \ldots$, first and second order contribution to displacement thickness defined in (5.5)

momentum thickness

$\Delta_1, \Delta_2 \ldots$ first and second order contributions to the momentum thickness defined in (5.6)

ϵ, a parameter defined as $\sigma^{-1/3}$

ζ, outer variable in low Prandtl number analysis defined by (3.18) or (4.41)

η, boundary layer variable defined as

$$= (2s)^{-1/2} \int_0^N \rho\, dN \quad \text{in Chapter 3}$$

$$= U_1 r^j N/(2\xi)^{1/2} \quad \text{in Chapter 4, 5}$$

θ, angle between axis of axisymmetric body and the tangent to meridian curve at any point (see Fig.2.1)

θ, a variable defined by equation (3.10)

$\tilde{\theta}$, inner variable in high σ flow defined by (4.48)

$\Lambda(s)$, principal thermal function defined as

$$= \frac{2(h_w - S_1)^{-1}}{U_1 r^{2j}} \frac{dh_w}{ds} \int_0^s U_1 r^{2j}\, ds$$

$\Lambda_d(s)$, principal displacement speed function defined as

$$= \frac{2}{U_1 U_2 r^{2j}} \frac{dU_2}{ds} \int_0^s U_1 r^{2j}\, ds, \text{ with } U_2 = U_2(s, 0)$$

$\Lambda_l(s)$, principal longitudinal curvature function defined as

$$= \frac{2}{k_l U_1 r^{2j}} \frac{dk_l}{ds} \int_0^s U_1 r^{2j}\, ds$$

$\Lambda_s(s)$, principal body shape function defined as

$$= \frac{2}{U_1 r^{2j+1}} \frac{dr}{ds} \int_0^s U_1 r^{2j}\, ds$$

$\Lambda_t(s)$, principal transverse curvature function defined as

$$= \frac{2}{k_t U_1 r^{2j}} \frac{dk_t}{ds} \int_0^s U_1 r^{2j}\, ds$$

$\Lambda_v(s)$, principal vorticity interaction function defined as

$$= \frac{2}{\alpha U_1 r^{2j}} \frac{d\alpha}{ds} \int_0^s U_1 r^{2j} ds$$

λ, displacement speed function defined as $= U_2/U_1$

μ, viscosity of the fluid

ξ, boundary layer variable defined as $= \int_0^s U_1 r^{2j} ds$

ρ, density of the fluid

σ, Prandtl number of the fluid defined as $= \mu C_p/k$

τ, skin friction

τ_1, τ_2, first and second order contributions to the skin friction defined in (5.3)

ϕ, outer variable in low pPrandtl number problem defined by (3.18)

χ, inner variable in high Prandtl number problem defined by (3.49)

ψ, stream function of the fluid

ψ_1, ψ_2, first and the second order stream functions in the inner (boundary layer) flow defined in (5.17)

Ψ_1, Ψ_2, first and the second order stream function in outer flow defined in (5.7)

Ω, external vorticity

Ω_1, Ω_2, first and second order vorticity in the outer flow defined in (5.7).

Superscripts;

', differentiation with respect to variable η

d, displacement speed

l, longitudinal curvature

S, stagnation enthalpy gradient

t, transverse curvature

v, external vorticity gradients.

Subscripts;

c, characteristic value

n, partial derivative with respect to n

N, partial derivatives with respect to N

r, adiabatic wall

s, partial derivative with respect to s

w, wall

ξ, partial derivative with respect to ξ

∞, free stream

1,i 1,ij universal functions of first order boundary layer

2,i 2,ij universal functions due to second order boundary layer.

LIST OF FIGURES

CHAPTER 5

SYNOPSIS

The present work deals with the study of the second order effects to the classical boundary layer theory when characteristic Reynolds number of the flow is only moderately large. However, in the beginning, we have studied some aspects of classical boundary layer theory itself, which have either not received sufficient attention in the literature or the treatment **is** not exhaustive or coherent.

The second order effects studied in the present work are due to transverse curvature, longitudinal curvature, external vorticity, stagnation enthalpy gradient and displacement, which are described by linear equations. For each of these second order effects a general theory of Görtler type is developed. The flow may be two dimensional or axisymmetric. Attention is focused on low speed and small temperature changes, so that the velocity field is essentially that of an incompressible fluid. We calculate temperature profile too, dissipation is neglected.

In addition to principal velocity and thermal functions a principal shape function is defined for axisymmetric flows in the study of first order boundary layer and a new principal function is introduced for each of the second order effects (i.e., principal transverse curvature function, principal longitudinal curvature function, principal vorticity interaction function and principal displacement speed function). These new

principal functions are of fundamental importance in determining the structure of the second order effects. In the present work we have studied the first two terms in the second order Görtler power series for the velocity and temperature profiles, by choosing the proper universal functions. The resulting system of total differential equations and the accompanying boundary conditions constitute the two point boundary value problems and are solved by Runge-Kutta method with Gill improvement on IBM 7044 Computer at Indian Institute of Technology Kanpur. Solutions are obtained for both accelerating and decelerating flows with Falkner-Skan pressure gradient parameter β_o between -0.198838 and 2.0 and the Prandtl number 0.7 to 3. The classical concepts of displacement and momentum thicknesses are extended to higher order boundary layer flows. Second order perturbation to displacement and momentum thicknesses are evaluated separately for each of the above mentioned second order effects. The results for the variation of various second order quantities such as changes in velocity profile, temperature profile, skin friction, heat transfer, displacement thickness and momentum thickness due to various second order effects are represented graphically. A critical comparison with various available results is presented.

Our solutions for the second order effects show that the transverse curvature increases the skin friction and heat transfer. The convex longitudinal surface curvature decreases the skin friction and heat transfer. The effect of external vorticity is to decrease the skin friction. The heat transfer is also

decreased except for $\beta_o > 1$ where it increases. The effect of stagnation enthalpy gradient is to decrease heat transfer. Finally the displacement speed increases the skin friction and heat transfer. Lastly, the results indicate that the convergence of Görtler power series for the first and the second order boundary layer functions becomes poor as favourable pressure gradient diminishes. Furthermore, in adverse pressure gradient (near $\beta_o = -0.198838$) some of the second order boundary layer quantities becomes very large compared with the corresponding first order boundary layer quantities. This indicates that the boundary layer (inner) expansion is not uniformly valid near the separation.

The other two problems studied in the beginning of present work are of first order boundary layer theory. First one is the study of compressible boundary layer with no pressure gradients, with arbitrary thermal properties (using the power law relations for the dependence of viscosity, density and Prandtl number upon temperature) separately for small, large and order unity values of Prandtl number. For limiting cases of low and high Prandtl number, the method of matched asymptotic expansions is used. In low Prandtl number problem it turns out that the outer flow is governed by a simple nonlinear equation of second order, which is correct to all orders in Prandtl number and can be solved once and for all. For the case of insulated wall this outer equation can be solved analytically. However, for heat transfer case it is solved numerically (to yield an outer solution which is also

correct to all orders in Prandtl number) by Runge-Kutta method. This numerical solution is matched asymptotically with the corresponding inner solution. For high Prandtl number case, the inner flow is governed by coupled nonlinear differential equations, with the outer boundary conditions to be determined by matching it with outer solution (which is singular) and, in general, is very difficult to solve exactly. In this case an approximate solution of the Weyl type is carried out. For Prandtl number of order unity, the Meksyn type method of inversion of variables is used. Further it is shown that if Prandtl number is constant, equal to unity the classical **Croccos** integral holds even with arbitrary properties. In all three cases of low, high and order unity values of Prandtl number, the leading terms are obtained for recovery factor and heat transfer rate at the wall. The results are displayed graphically.

The second problem in the study of first order boundary layer is the study of heat transfer in Falkner-Skan flows for small, large and order unity values of Prandtl number (dissipation is included in the energy equation). For low and high Prandtl numbers, the method of matched asymptotic expansions is used while for Prandtl number of order unity a Meksyn type method is used. The problem has been studied extensively in the past and most of the authors have either ignored the pressure gradient or dissipation or both.

In low and high Prandtl number flows, many higher order perturbations are obtained and results are good even up to Prandtl number of order unity. Detailed examination of the results indicate that in favourable pressure gradient the results of both the analyses put together fill the gap in the intermediate region where Prandtl number is of order unity. In adverse pressure gradient, there is a gap in the intermediate region $0.1 < \sigma < 1$ due to poor convergence of both the series in the above region. However, if the series are properly Eulerised results fill the gap.

Thus we see that in some sense it suffices to analyse the two limiting cases: of low and high Prandtl numbers, provided large number of terms are computed in each expansion and series are properly Eulerised.

CHAPTER 1

INTRODUCTION

The classical boundary layer theory of Prandtl represents the leading term in an asymptotic expansion of the Navier-Stokes equations for high Reynolds number R, which becomes asymptotically exact only for very large Reynolds number. Nevertheless, these asymptotic solutions are exceedingly useful from a practical point of view owing to the fact that in many cases these have been found to hold with surprising accuracy even under distinctly nonasymptotic conditions. It appears desirable therefore to study a general technique for constructing the corrections to these asymptotic solutions in order to extend their range of validity. The practical utility of such a refinement is suggested by an estimate of Lagerstrom and Cole (1955) that the second approximation may in some cases predict the results accurately down to Reynolds number ten or even five. Moreover, it is shown here that in some cases (see Chapter 4) it suffices in some sense to study the two limiting cases of very small and very large values of the parameter, as the results obtained from above two cases overlap in the intermediate range, where parameter is of order unity.

Further, the Prandtl boundary layer theory does not account for what Rott and Lenard (1959) call the secondary effects. These effects may be classified as those due to longitudinal curvature,

transverse curvature, external vorticity gradients, stagnation enthalpy gradients, displacement, slip and temperature jump. At lower Reynolds numbers, however, these various (above mentioned) effects are of the order $R^{-1/2}$ and necessiate corrections to the classical boundary layer theory. However, for flow situations in which these effects are of order unity (for example if $KR^{-1/2}$ is of order unity which implies K the longitudinal curvature is of the order $R^{1/2}$ as R becomes very large), some authors have modified the tangential momentum equation of the Prandtl so as to include such terms but have ignored the fact that a corresponding change becomes necessary in the normal momentum equation too — $\partial p/\partial n$ will now be of the order of $R^{1/2}$ and hence the surface pressure cannot be equated to its value at the edge of boundary layer. This statement may appear too strong, but we feel that under these circumstances the treatment should be called empirical rather than one based on Prandtl's boundary layer theory. Furthermore, these authors have studied only the individual effects by formulating a nonlinear total differential equation. Thus in a problem dealing with various effects simultaneously we cannot use the results of the above authors as the superposition is not valid. In the present work only those flows will be studied in which these effects are really secondary and appear only in the form of corrections (described by linear equations) to the classical (first order) boundary layer.

A systematic formulation of the scheme to calculate the higher order corrections to classical boundary layer theory is given by Van Dyke (1961, 1962) for compressible and constant properly fluids respectively. The method used is the technique of matched asymptotic expansions developed by Kaplun (1954, 1967), Lagerstrom and Cole (1955), etc. A feature of the perturbation technique is that the resulting second (and higher) order boundary layer (inner) and outer equations are linear. Following the suggestion of Rott and Lenard (1959), we can subdivide these higher order problems into a number of simpler problems, each of which has a clear physical interpretation and then superpose.

The second order boundary layer theory developed for compressible flow of a fluid by Van Dyke (1961), is used by him to discuss the flow past an axisymmetric stagnation point. He uses the first term of the Blasius series expansion in the coordinate along the body and finds the solution of the first and second order boundary layer equations in the stagnation region with viscosity proportional to temperature, Prandtl number is 0.7 and the ratio of wall to stagnation temperature is 0.2. Davis and Flügge-Lotz (1964a) have further calculated the vorticity effect (due to entropy gradients only) for various values of the ratio of wall to stagnation temperature when viscosity varies as the square root of temperature and the Prandtl number is 0.71. The Davis and Flügge-Lotz (1964b) have solved the partial differential equations of the second order boundary layer theory with the aid of finite

difference method for studying the second order effects (except stagnation enthalpy gradients) for hypersonic flow past a paraboloidal and hyperboloidal at free stream Mach number of infinity, and a sphere at free stream Mach number of ten, with Prandtl number 0.71. The corresponding problems of boundary layer around a circular cylinder and slab with circular leading edge have been analysed by Fannelöp and Flügge-Lotz (1966).

For the case of constant property flow Van Dyke (1962b) has studied the flows past the stagnation point of axisymmetric or plane body and also of cusped leading edges at ideal incidences. He also studied the heat transfer with Prandtl number 0.7 and 1.0. Later Van Dyke (1964) applied the above theory to a parabolic cylinder in uniform stream. Devan and Oberai (1964) formulated an integral relation of the Karman type for the second order momentum boundary layer, the flow being two dimensional. Later Devan (1965) formulated the integral relation in Crocco form (for the above problem) and applied the Dorodnitzyn method of approximate solutions.

In addition to Van Dyke, the second order boundary layer theories have been given by Lenard (1961) and Malsen (1963). These authors have provided the solutions which are valid in the stagnation region only. In general, they are in agreement with Van Dyke in their results with the exception of the solution for vorticity interaction. Honda and Kiyokowa (1969) have also developed a systematic second order boundary layer theory for the flow of an incompressible fluid in stream function coordinates.

A critical examination of the various second order and related theories and comparison with experiments up to that time have been discussed rather thoroughly by Van Dyke (1962c). Cheng (1965) has also given a good account of various second order theories. Furthermore, each of the secondary effect has been dealt with by several authors. The results of most of them are in disagreement with each other. One source of confusion was the lack of realization that all the second order effects are not independent in the strict sense; the difficulties resulting from the arbitrary division of these effects can be avoided only by treating them concurrently. However, whatever the division, the effects relegated to the category of displacement are global in nature (and as such are most difficult to calculate) while the others are local. Second source of confusion was that the analysis of most of the authors was not systematic upto the second order as they have omitted some of terms of the same order as those retained. Thirdly, in some cases the matching of the inner and outer flow has not been carried out correctly.

The subject of longidutinal curvature has been studied by Tani (1949, 1954), Murphy (1953, 1962, 1965), Yen and Toba (1961, 1962), Hayasi (1963), Massey and Clayton (1965), Schultz-Grunow and Breuer (1965) and Narasimha and Ojha (1967). The results previous to Van Dyke are controvertial. Thus Tani and Murphy concluded that the skin friction will decrease on the convex side because of the longitudinal curvature, Yen and Toba arrive at just the opposite conclusion. However, the formulations of Murphy,

Yen and Toba and Hayasi are not consistent up to the order considered as these authors have omitted some of the terms of the second order. Schultz-Grunow and Breuer studied the effect of longitudinal curvature with no pressure gradient by formulating a nonlinear differential equation. Their equation contains all terms of order unity and $R^{-1/2}$ present in Van Dyke's equation and include in addition one viscous term of order R^{-1}. The above analysis is not systematic in the sense that using their formulation one can not obtain higher order approximations. Similar comments apply to the work of Massey and Clayton who also constructed a nonlinear equation. Narasimha and Ojha using Van Dyke's formulation studied only the similarity term for the effect of longitudinal curvature.

The transverse curvature problem for a longitudinal flow of an incompressible fluid over a circular cylinder has been studied by Seban and Bond (1951), Glauert and Lighthill (1955), Stewartson (1955) and Jaffe and Okamura (1968). For compressible flow of a fluid past a slender body of revolution, the transverse curvature effects are studied by Probstein and Elliot (1956) and Yasuhara (1956). Further, Stewartson (1964), Li and Gross (1964) and Ellinwood and Mirels (1969) have also studied the transverse curvature effects in hypersonic flow. All the above authors have formulated the problem by including the transverse curvature terms in the first order equations.

The effect of external vorticity on the boundary layer was first pointed out by Ferri and Libby (1954). The first investigators

of the vorticity effect Li (1955, 1956) and Glauert (1957, 1962) engaged in a prolonged arguments, whether or not the pressure gradient on the boundary layer is altered. Hayes (1956) argued that at hypersonic speeds the external flow has to pass through a strong shock wave, and, therefore may contain vorticity comparable to that in the boundary layer. He studied the vorticity interaction in such a flow by modifying the outer boundary condition in the classical boundary layer theory. With the assumption that the boundary layer displacement does not appreciably change the pressure and velocity distribution along a streamline in the outer flow, he arrives at a modified outer boundary condition for the velocity field f_η in the form

$$f_\eta^2 = 1 + 2\Omega f , \eta \to \infty,$$

where Ω is the vorticity interaction parameter, defined as, the ratio of the external vorticity to some average value of vorticity in the boundary layer. Kemp (1959) used the above condition to study the vorticity interaction. Here also, we wish to comment that such an analysis is empirical as mentioned in second para. Rott and Lenard (1959) have invariably disregarded the pressure change associated with vorticity, which is immertial for high cooled wall (with which these investigators were mainly concerned) because the effect is then slight. However, in general, this induced pressure gradient should be accounted for while studying the vorticity interaction. Ferri, Zakkay and Ting (1961) studied the problem on the basis of local flat plate similarity. In particular, they follow Lees (1956) in altogether neglecting the

tangential pressure gradient and external vorticity. Although the neglect pressure gradient has been shown to yield reasonable accuracy for the first order boundary layer on a cooled wall, the significant role of pressure in the second order theory would suggest careful scrutiny the results. Further, their procedure of joining the inner and outer solutions for velocity and temperature profiles involves a complicated patching process along two different lines, rather than matching them asymptotically. Cheng (1961) has also studied the vorticity effect by using the thin shock layer approximation.

Prandtl (1935) himself suggested, how the boundary layer on a flat plate might be corrected by the effects of displacement. Massey and Clayton (1965) in their longitudinal curvature analysis also include a consideration of displacement effect by formulating a nonlinear differential equation by a procedure which amounts to matching the outer limit of inner solution at the edge of boundary layer. This clearly an oversimplification. The displacement effects are more difficult to calculate than others. The reason is that they are global in nature, being dependent upon the classical boundary layer solution in the whole range and its influence upon the outer inviscid solution, where as the other effects are local. To be exact the effect of displacement thickness should be solved by perturbing the first order inviscid solution. However, in practice the flow due to displacement thickness is studied by

solving the first order inviscid problem for a fictitious body with appropriate boundary conditions. The profile of this fictitious body is obtained by a suitable fit to the original body and its displacement thickness. Thus Lenard (1961) has suggested approximating a sphere (locally near the stagnation point) and its displacement thickness by a slightly larger concentric sphere. Malson (1961) proposes using a dilated and shifted sphere. Davis and Flügge-Lotz (1964b), Fannelöp and Flügge-Lotz (1966) and Honda and Kiyokowa (1969) have also approximated their bodies and displacement thicknesses by such fictitious bodies.

In global problems involving shock waves, the flow in the shock layer, i.e., the region between the shock wave and the body is generally divided into an inviscid (outer) region and a viscous (inner) boundary layer near the body, and is solved according to the following procedure. We first calculate the first order inviscid flow using the Rankine-Hugoniot relations as the outer boundary conditions, and then solve the classical boundary layer equations using the boundary conditions at the wall, matching it with first order inviscid solution. Next, for the second order inviscid flow (due to displacement) we still use the Rankine-Hugoniot relations as the outer boundary conditions. This displacement flow, in turn determines the second order boundary layer. Thus the second order boundary layer depends indirectly upon the Rankine-Hugoniot relations. The next higher order, i.e., the third order approximation to the outer flow requires that for the outer boundary condition,

the Rankine-Hugoniot relations be corrected due to curvature of shock wave (these corrections involve the study of the shock wave structure). The modified shock conditions and the associated analysis of the shock transition zone have been given by Ting and Chow (1961) and Germain and Guiraud (1962). Oberai (1963) has calculated the corrections in the Rankine-Hugoniot relations in the stagnation region for the case of sphere in hypersonic flow. Further with the constant density assumption he has studied the third order effects in outer flow. Finally this outer flow will determine the third order boundary layer. Kao (1964) has calculated the effect of these corrections on the third order boundary layer.

Due to recent interest in high speed flights at high altitudes (for example, a space vehicle entering in the earth atmosphere encounters a wide range of flow conditions varying from free molecular flow at high altitudes to continuum at low altitudes) some authors attempted to fill the gap between the continuum and free molecular flow regimes. The solution to describe the entire (transitional) regime between continuum and free molecular flow, though very desirable, was foreseen to be a formidable task. So far the efforts have been made to obtain solutions for the following sub-regimes. (1) Near free molecular flow: here, the procedure adopted has been to include first order collisions, i.e., collision with a free stream molecule by one that has already hit the surface. (2) Near continuum: here, the (empirical) approach is that the governing

equations are those of the continuum regime, but the boundary conditions near the wall are velocity slip and temperature jump as obtained from kinetic theory of gases. It is well-known that the velocity slip and temperature jump are proportional to the mean free path or in some sense to R^{-1}. If the results of the two above analyses, put together could fill the gap the problem is completely solved. However, the range of validity of the solutions in these subregimes has not been definitely established.

It is to be recognised that the final justification for the continuum analysis has to come from the study of the Boltzmann equation. In classical Chapman-Enskog expansion of the solution of Boltzmann equation (ref, Chapman and Cowling 1939) the leading term gives the inviscid Euler equations, the first two together provide the Navier-Stokes equations and the first three the Burnett equations. Though it would be natural to expect that the Burnett equations will provide better results than those by Navier-Stokes, it is found that in many situations the results obtained by the use of Navier-Stokes equations compare better with the experiments than those by the Burnett equations; in some problems the use of Burnett equations gives anomalous solutions. Stewartson (1964) has observed that, provided the temperatures are not extreme and the Kundsen number is small, there is no known case of disagreement between experiment and theory which can be traced to a fundamental

deficiency in the Navier-Stokes equations. Even the flow properties inside the moderately strong shocks are given fairly accurately by the Navier-Stokes equations, even though their thickness is only a few mean free path and the corresponding Kundsen number not very large. Of significance in this respect is the recent study of shock transition zone by Liepmann, Narasimha and Chahine (1962) in which Boltzmann equation with Krook's collision model is used. The solutions obtained show small differences from Navier-Stokes solution on the high pressure side of the shock even for a Mach number of ten. The convergence of the Chapman-Enskog expansion has not been established. Moreover, it appears that the full depth of the properties of Navier-Stokes equations have not been plumbed and that their usefulness extends beyond the theoretical range. Thus it is preferable to regard the Navier-Stokes equations as axiomatic with the constitutive properties of the fluid to be determined from experiments. Then develop the theory from the Navier-Stokes equations in as rigorous way as possible. The solution of Boltzmann equation (even for simple situation) is a formidable task; so the further comparison with experiments and physical observations can help us to decide the validity of above axiom.

The main aim of the present work is to develop a general theory of the Görtler (1957) type for higher order boundary layer flows past ~~a~~ plane or axisymmetric bodies. In the spirit of what has been said earlier we shall be studying only the second order

effects due to longitudinal curvature, transverse curvature external vorticity gradients, stagnation enthalpy gradient and displacement. Attention will be focused on low speed and small temperature gradients, so that the velocity field is essentially that of an incompressible fluid. We calculate the temperature profile too, (dissipation is ignored). We solve for first two terms in the second order Gortler power series for velocity and temperature profile, by choosing the proper universal functions. The resulting system of total differential equations and the accompanying boundary conditions constitute the two point boundary value problems, and are solved by Runge-Kutta-Gill method on IBM 7044 computer at Indian Institute of Technology, Kanpur. Solutions are obtained for both accelerating and decelerating flows with Falkner-Skan pressure gradient parameter between -0.198838 and 2 and the Prandtl number 0.7 to 3. Various other boundary layer characteristics such as displacement and momentum thicknesses etc. are also studied.

However, at the beginning we have studied two problems of the first order boundary layer, which have either not received sufficient attention in the literature or the treatment is not exhaustive or coherent. First is the study of first order compressible boundary layer with no pressure gradient (using the power law relations for the dependence of viscosity, density and Prandtl number upon temperature) separately for the small, large and of order unity values of the Prandtl number. The

problem of low Prandtl number is of interest in liquid metals and plasma. For air the Prandtl number is 0.73, but the flights at high speeds and altitudes lead to high temperature and low pressure at which the Prandtl number can became very small (see inset in Fig.3.6 after Mitchell 1962). The high Prandtl number problem is of interest in liquids and vapours. It is shown in Chapters 3, 4 that the study of these limiting cases of small and large Prandtl numbers is of great use even under distinctly non-asymptotic conditions.

The general problem of compressible laminar boundary layer has been studied extensively in the past and a good review is given by Stewartson (1964). In general, it is found difficult to account for arbitrary fluid parameters and Mach number, and most of the authors assumed the Prandtl number of order unity. Except, in specific numerical calculations, accounts of which are given by several authors, notably Kuerti (1950), Young (1953), Van Driest (1959), Moore (1964) and Stewartson (1964); in most of the analyses it is assumed that the viscosity of the fluid varies linearly with the temperature. This is true from kinetic theory of gases, only, if the gas is made up of the so called Maxwell molecules obeying the inverse fifth power law of repulsion. Most of the gases, however, are not Maxwellians. The viscosity - temperature relationship is better represented by simple power law $\mu \propto T^{w}$ where w is a constant usually between 1/2 and 1. Though the Sutherland law can also be employed when the interest is limited to a limited range of temperature.

Here, we have used the method of matched asymptotic expansions for the low and high Prandtl number problems. In low Prandtl number problem it turns out that the outer flow is governed by a simple nonlinear equation of the second order, which is correct to all orders in Prandtl number. In solving this outer equation one can proceed in the routine way by expanding the outer variable in powers of Prandtl number. Because of certain simplifying features (see, Section 3.3) which enables us to adopt a most efficient technique, i.e., we expand the arbitrary constant of intigration instead of solution. For the case of insulated wall this outer equation can be solved analytically. However, for the heat transfer case it is solved numerically by Runge-Kutta method. For high Prandtl number problem the inner flow is governed by coupled nonlinear differential equations, with the outer boundary conditions to be determined by matching it with the inner limit of outer solution (which is singular) and, in general, is very difficult to solve exactly. In this case an approximate solution of the Weyl type is carried out. For Prandtl number of order unity we use a certain asymptotic method developed by Meksyn (1961). In all above three cases of low, high and order unity Prandtl numbers, the leading terms are obtained for the recovery factor and the heat transfer rate at the wall.

In the second problem, we have studied the heat transfer in the Falkner-Skan flows when dissipation is included in

the energy equation. The problem has been studied extensively in the literature but most of the authors have either ignored the pressure gradient, the dissipation or both. The high Prandtl number flow is studied by Narasimha and Vasantha (1966) by ignoring the pressure gradient, and by Lighthill (1950), Adams (1962) and by Goddard and Acrivos (1966) by neglecting dissipation. In general, high Prandtl number case leads to a singular perturbation type problem; however, if dissipation is neglected, the problem becomes one of regular perturbation type. The problem of low Prandtl number is studied is studied by Adams (1962), Goddard and Acrivos (1966) by ignoring the dissipation and Morgan, Pipkin and Warner (1958) by neglecting that portion of the dissipation which is due to compressibility and by Afzal (1967) by neglecting the pressure gradient. For Prandtl number of order unity, the problem is studied by Merk (1959) and Evans (1967). In the present work, we have used the method of matched asymptotic expansions for the low and high Prandtl number problems. For Prandtl number of unity the Meksyn type method is employed.

In the low and high Prandtl number flows, many higher order perturbations are calculated and the results obtained are good even up to Prandtl number of order unity, so the results of both the low and high Prandtl number analyses overlap. Thus in a sense it suffices to analyse the two limiting cases: of low and high Prandtl numbers.

CHAPTER 2

FORMULATION OF SECOND ORDER BOUNDARY LAYER THEORY

2.1 Introduction:

We now describe a brief resume of the derivation of the scheme for calculating the higher order corrections to the classical boundary layer theory, designed for unreparated flows. We shall follow the treatment of Van Dyke (1962). It consists of in developing the general second order boundary layer theory, the governing equations for which are obtained from Navier-Stokes equations by an expansion procedure in inverse power of the square root of Reynolds number. The procedure used is the technique of matched asymptotic expansions developed by Kaplun (1954, 1967), Lagerstrom and Cole (1955), etc. In this method we generally study the inner and outer limits of the equations and try to match their solutions in an overlap domain (see, for example Van Dyke 1964, Kaplun 1967, Cole 1968). The inner and outer expansions (obtained by repeated application of these limit processes) result in replacing the Navier-Stokes equations by two separate sets of equations, one set which is valid in an outer region and other valid in the inner viscous region (near the wall in the problems discussed here). It is well known that none of the above two sets of equations are determinate in the sense that the number of boundary conditions are not enough.

The additional boundary conditions required to make the system closed are obtained by what Kaplun (1957) calls the extension theorem.

In the spirit of what has been said in the first chapter we shall be considering only the low speed flows with small temperature differences (so that the heat added by conduction and dissipation are small thereby the density is sensibly constant, as are the transport properties). The governing equations for such a flow are Navier-Stokes and the corresponding energy equation. Furthermore, we shall be studying only plane or axisymmetric flow past a semi-infinite body described by an analytic curve.

2.2 Coordinate System:

We shall employ an orthogonal coordinate system (s, n) shown in Fig. 2.1, where s is the distance along the surface (measured either from the stagnation point or the leading edge) and n is the distance normal to surface. The corresponding velocity components of the velocity vector $\underset{\sim}{U}$ are (u, v). Let K(s) be the curvature of the body section reckoned positive for a convex body. In axisymmetric flow, $\theta(s)$ is the angle between the axis and the tangent to the meridian curve at any point and r(s) is the distance of the point from axis. It is to be noted that all these quantities are not independent, but are connected according to

$$\sin\theta = dr/ds, \qquad \cos\theta = -K^{-1}\, d^2r/ds^2. \qquad (2.1)$$

The length element dl in space is now given by

$$dl^2 = (1 + Kn)^2 \, ds^2 + dn^2 + (r + n \cos \theta)^{2j} \, d\phi^2. \qquad (2.2)$$

Here $j = 0$ for plane flow and $j = 1$ for axisymmetric flow, and ϕ is the cylindrical coordinate (linear distance normal to the plane of flow in two dimensional flow and the azimuthal angle in the axisymmetric flow).

2.3 Governing equations:

The basic equations of motion for a steady plane or axisymmetric flow in the present coordinates are given in Van Dyke (1961). These are quite lengthy and will not be given here. Alternatively, we shall be using vector notation. The basic equations of motion, for such a flow of constant property fluid can be written, in nondimensional vector form (here all distances $\underset{\sim}{X}$ are nondimensionalised with respect to characteristic length L of the body, velocity $\underset{\sim}{U}$ and enthalpy h by the characteristic reference values U_c and h_c, and pressures P by ρU_c^2) as

$$\operatorname{div} \underset{\sim}{U} = 0, \qquad (2.3)$$

$$\underset{\sim}{U} \cdot \operatorname{grad} \underset{\sim}{U} + \operatorname{grad} P = -R^{-1} \operatorname{curl} \operatorname{curl} \underset{\sim}{U}, \qquad (2.4)$$

$$\underset{\sim}{U} \cdot \operatorname{grad} (h + EU^2/2) - R^{-1} \nabla^2 (\sigma^{-1} h + EU^2/2)$$
$$= ER^{-1} [(\operatorname{grad} \underset{\sim}{U}) \cdot (\operatorname{grad} \underset{\sim}{U})^*], \qquad (2.5)$$

where we have written R for the Reynolds number $(= \rho U_c L/\mu)$, σ for the Prandtl number $(= \mu C_p/k)$ and E for Eckert number $(= U_c^2/h_c)$. Here μ is the coefficient of viscosity, ρ the density, k the thermal conductivity and C_p the specific heat at constant pressure.

Some times it is preferable to reduce the momentum equation (2.4) to two scaler equations which are the generalisations of Bernoulli's and Kelvin's theorems. First forming the dot product of (2.4) with $\underset{\sim}{U}$, we get,

$$\underset{\sim}{U} \cdot \text{grad } (P + U^2/2) = R^{-1} \underset{\sim}{U} \cdot \nabla^2 \underset{\sim}{U}, \tag{2.4a}$$

where left hand side is the rate of change of total pressure $P + U^2/2$ along a stream line. Taking the curl of (2.4), we get,

$$(r + n \cos \theta)^j \underset{\sim}{U} \cdot \text{grad}\left[\Omega (r + n \cos \theta)^{-j} \right] \tag{2.4b}$$

$$= R^{-1} \text{ div} \left[(r + n \cos \theta)^{-j} \text{ grad}\left\{ \Omega (r+n \cos \theta)^j \right\} \right]$$

here again the left hand side is the rate of change of vorticity Ω along a stream line (it is recalled that for plane or axisymmetric flow the vorticity Ω has only one component which is always perpendicular to the section).

The boundary condition at the solid surface for the velocity profile (v_w is the speed of suction or injection of the fluid and $\hat{n}$ is the unit vector in n direction) is

$$\underset{\sim}{U} = \hat{n} \, v_w(s) \tag{2.6a}$$

which is the famous no slip condition for permeable surface, ?
and for enthalpy profile is

$$h = h_w(s) \quad \text{or} \quad \partial h/\partial n = 0. \tag{2.6b}$$

Only one of the two conditions at the surface in (2.6b) is required, the first prescribes the surface enthalpy to $h_w(s)$ and the second implies an insulated surface. For upstream the

flow is to approach prescribed (possibly nonuniform) velocity and enthalpy fields U_∞ and h_∞ respectively.

2.4 Formulation:

The problem defined by equations (2.3), (2.4), (2.5) and (2.6) is highly complex and nonlinear. It is fore seen to be a formidable task to find the solution for entire range of Reynolds number. So far, the efforts have been made to obtain solutions for the limiting cases. We shall study the boundary layer flow because this provides a natural small parameter $R^{-1/2}$ for studying the asymptotic solutions at large Reynolds number by the method of matched asymptotic expansions.

2.4.1 Outer Expansion:

Following Kaplun (1954), we define an outer limit (Euler limit) as the process $R \to \infty$ with $\underset{\sim}{X}$ fixed. A repeated application of the outer limit process in conjunction with appropriate sequence of gauge functions gives the outer expansion. In general the outer expansion is straight forward. For unseparated flows past analytic bodies, it has been found that the sequence of gauge functions consist of inverse power of square root of Reynolds number. The outer expansion therefore, has the form,

$$\begin{aligned}
\underset{\sim}{U}(\underset{\sim}{X};\ R) &= \underset{\sim}{U}_1(\underset{\sim}{X}) + R^{-1/2}\,\underset{\sim}{U}_2(\underset{\sim}{X}) + \ldots \\
P(\underset{\sim}{X};\ R) &= P_1(\underset{\sim}{X}) + R^{-1/2}\,P_2(\underset{\sim}{X}) + \ldots \\
h(\underset{\sim}{X};\ R) &= H_1(\underset{\sim}{X}) + R^{-1/2}\,H_2(\underset{\sim}{X}) + \ldots \\
\Omega(\underset{\sim}{X};\ R) &= \Omega_1(\underset{\sim}{X}) + R^{-1/2}\,\Omega_2(\underset{\sim}{X}) + \ldots \\
\psi(\underset{\sim}{X};\ R) &= \Psi_1(\underset{\sim}{X}) + R^{-1/2}\,\Psi_2(\underset{\sim}{X}) + \ldots .
\end{aligned} \tag{2.7}$$

Here ψ is the stream function, defined in usual way from equation of continuity (2.3). We now substitute the outer expansion (2.7) in equations (2.3), (2.4) and (2.5), and collecting the coefficients of the various powers of R give the equations for successive outer approximations. Thus the first order equations,

$$\operatorname{div} \underset{\sim}{U}_1 = 0 \tag{2.8a}$$

$$\underset{\sim}{U}_1 \cdot \operatorname{grad} \underset{\sim}{U}_1 + \operatorname{grad} P_1 = 0 \tag{2.8b}$$

$$\underset{\sim}{U}_1 \cdot \operatorname{grad} (H_1 + EU_1^2/2) = 0 \tag{2.8c}$$

are the inviscid Euler equations of motion, and the second order equations are

$$\operatorname{div} \underset{\sim}{U}_2 = 0 \tag{2.9a}$$

$$\underset{\sim}{U}_1 \cdot \operatorname{grad} \underset{\sim}{U}_2 + \underset{\sim}{U}_2 \cdot \operatorname{grad} \underset{\sim}{U}_1 + \operatorname{grad} P_2 = 0 \tag{2.9b}$$

$$\underset{\sim}{U}_1 \cdot \operatorname{grad} (H_2 + E\underset{\sim}{U}_1 \cdot \underset{\sim}{U}_2) + \underset{\sim}{U}_2 \cdot \operatorname{grad} (H_1 + EU_1^2/2) = 0. \tag{2.9c}$$

These are the displacement flow equations and are also inviscid in the sense that there do not contain the viscous term (of order R^{-1} in outer approximation).

It is well known that Euler (outer) equations fail at the surface, through loss of the highest order derivative in the Navier-Stokes equations, in that, these can not satisfy the no slip condition. The outer expansion is invalid with in a distance of the order $R^{-1/2}$ from the surface. To overcome this difficulty we need a different approximation, what Kaplun calls the inner limiting process; i.e., the Prandtl boundary

layer type which is valid in the region of order $R^{-1/2}$ near the body. Also it is observed by Lagerstrom (1964) and Kaplun (1967) that for a given configuration the Euler solution is not unique. The question is than which Euler solution for a given geometry of flow is the Euler (outer) limit of the Navier-Stokes equations, unless we can guess it directly, finding the relevant solution would presuppose a knowledge of the solutions of Navier-Stokes equations for infinite R; but this is exactly the solution to be found by perturbation procedure. The potential flow solution of Euler equation (also satisfies Navier-Stokes equations) is only one such possible solution. If the on coming stream contains vorticity, a solution of Euler equation does not in general satisfy the Navier-Stokes equations. However, it does so in special cases, for example in plane flow with constant vorticity and axisymmetric flow with vorticity proportional to radius. In other words out of a family of solutions of Euler equation we choose one which satisfies the far upstream boundary conditions and at the same time matches with the corresponding inner solution (term matching will be discussed later in detail). Furthermore, the serious difficulties arise in finding the Euler solution for non-analytical shapes (see, Goldstein 1960). For separated flows the appropriate Euler solution is not known and for unseparated flows past finite bodies it has a complicated mathematical description. This motivates the restriction of our study to unseparated flow past a semi-infinite body described by an analytic curve.

In the present work we shall not study the solution of the outer equations, assume that the first and the second order outer equations (2.8) and (2.9) with inner boundary conditions determined by matching have been solved (numerically or analytically), which provides at least, say, the slip velocities at the wall $U_1(s, 0)$ and $U_2(s, 0)$ respectively. Now the pressure distribution etc. at the wall are found from the study of kinematic equations (2.4a) and (2.4b). Substituting (2.7) in equations (2.4a) and (2.4b) and collecting the like powers of R, the first order terms are

$$\underset{\sim}{U}_1 \cdot \text{grad}\ (P_1 + U_1{}^2/2) = 0 \qquad (2.10a)$$

$$\underset{\sim}{U}_1 \cdot \text{grad}\left[\Omega_1 (r + n \cos \theta)^{-j}\right] = 0. \qquad (2.10b)$$

Now each of the equations (2.10a), (2.10b) and (2.8c) represents the conservation of total pressure, vorticity and total enthalpy respectively, along the first order stream lines described by Ψ_1 (obtained from first order continuity equation 2.8c) Further, the equations (2.10a) and (2.10b) are connected by Bernonlli equation (as a consequence of Euler equations 2.8),

$$\Omega_1 (r + n \cos \theta)^{-j} = - d(P_1 + U_1{}^2/2)/d\,\Psi_1 .$$

These equations (2.10a), (2.10b) and (2.8c) are integrated to give,

$$P_1 + U_1{}^2/2 = B_1(\Psi_1) \qquad (2.11a)$$

$$\Omega_1 (r + n \cos \theta)^{-j} = - dB_1(\Psi_1)/d\,\Psi_1 \qquad (2.11b)$$

$$H_1 + EU_1{}^2/2 = S_1(\Psi_1) \qquad (2.11c)$$

where $B_1(\Psi_1)$ and $S_1(\Psi_1)$ are the Bernoulli and total enthalpy functions to be evaluated from upstream conditions. The second order kinematic equations are,

$$\underset{\sim}{U}_1 \cdot \text{grad}\ (P_2 + \underset{\sim}{U}_2 \cdot \underset{\sim}{U}_1) + \underset{\sim}{U}_2 \cdot \text{grad}\ (P_1 + U_1^2/2) = 0 \qquad (2.12a)$$

$$\underset{\sim}{U}_1 \cdot \text{grad}\left[\Omega_2 (r + n \cos\theta)^{-j}\right] + \underset{\sim}{U}_2 \cdot \text{grad}\left[\Omega_1 (r + n \cos\theta)^{-j}\right] = 0. \qquad (2.12b)$$

Similar comments apply to the equations (2.12a), (2.12b) and (2.9c), i.e., the flow quantities (total pressure etc.) are actually conserved along perturbed stream lines rather than the first order streamlines. When the on coming stream is prescribed, i.e., independent of Reynolds number the first integral of (2.12a), (2.12b) and (2.9c) is given by,

$$P_2 + \underset{\sim}{U}_2 \cdot \underset{\sim}{U}_1 = \Psi_2 B_1'(\Psi_1) \qquad (2.13a)$$

$$\Omega_2 (r + n \cos\theta)^{-j} = -\Psi_2 B''_1 (\Psi_1) \qquad (2.13b)$$

$$H_2 + E\, \underset{\sim}{U}_2 \cdot \underset{\sim}{U}_1 = \Psi_2 S'_1(\Psi_1). \qquad (2.13c)$$

Now the pressure, vorticity and enthalpy at the wall from the first order equations (2.11) are

$$P_1(s, 0) = B_1(0) - U_1^2(s, 0)/2 \qquad (2.14a)$$

$$\Omega_1(s, 0)\ r^{-j} = -B'(0) \qquad (2.14b)$$

$$H_1(s, 0) = S_1(0) - EU_1^2(s, 0)/2 \qquad (2.14c)$$

and from the second order equations (2.13) we get,

$$P_2(s, 0) = B_1'(0)\Psi_2(s, 0) - U_1(s, 0)\ U_2(s, 0) \qquad (2.15a)$$

$$\Omega_2(s,\ 0)\ r^{-j} \ = \ -\ B_1''(0)\,\Psi_2(s,\ 0) \qquad (2.15b)$$

$$H_2(s,\ 0) \ = \ S_1'(0)\,\Psi_2(s,\ 0)\ -\ EU_1(s,\ 0)\ U_2(s,\ 0). \qquad (2.15c)$$

Here $U_1(s, 0)$ and $U_2(s, 0)$ are known from the solution of outer equations and $\Psi_2(s, 0)$ will be determined later.

Further, we shall be needing the normal derivatives of the outer solution at the surface. Evaluating the equations (2.8a) and (2.8b) at the surface gives two of the required normal derivatives,

$$V_{1n}(s,\ 0) \ = \ -r^{-j}\ d\left[r^j\ U_1(s,\ 0)\right] /\ ds \qquad (2.16a)$$

$$P_{1n}(s,\ 0) \ = \ K\ U_1^{\,2}(s,\ 0). \qquad (2.16b)$$

The other normal derivatives are found by first differentiating (2.11a) and (2.11c) with respect to n and then evaluating at the surface (using the definition of $\Omega_{1n}(s, 0) = r^j U_1(s, 0)$ from 2.8a) we obtain,

$$U_{1n}\ (s,\ 0) \ = \ -\ KU_1(s,\ 0)\ +\ r^j\ B_1'(0) \qquad (2.16c)$$

$$H_{1n}\ (s,\ 0) \ = \ U_1(s,\ 0)\left[r^j\ S_1'(0)\ -\ r^j E\ B_1'(0) + EKU_1\ (s,\ 0)\right]. \qquad (2.16d)$$

2.4.2. Inner expansion:

It is now clear that a different inner limiting process is to be constructed, which is valid in the region of order $R^{-1/2}$ near the surface. Following Prandtl we introduce the inner variable,

$$N \ = \ R^{1/2} n$$

and study an inner (Prandtl) limit as the process $R \to \infty$ with N fixed. Again the repeated application of Prandtl limit process along with appropriate sequence of gauge functions will produce an inner expansion. For present case the appropriate inner expansions are,

$$\begin{aligned} u(s, N; R) &= u_1(s, N) + R^{-1/2} u_2(s, N) + \ldots \\ v(s, N; R) &= R^{-1/2} v_1(s, N) + R^{-1} v_2(s, N) + \ldots \\ P(s, N; R) &= p_1(s, N) + R^{-1/2} p_2 (s, N) + \ldots \qquad (2.17) \\ h(s, N; R) &= h_1(s, N) + R^{-1/2} h_2 (s, N) + \ldots \\ \psi(s, N; R) &= R^{-1/2} \psi_1(s, N) + R^{-1} \psi_2(s, N) + \ldots \end{aligned}$$

Substituting these expansions in equations (2,3), (2.4) and (2.5) and collecting the coefficients of like powers of R; we get for the first order approximation

$$(r^j u_1)_s + (r^j u_1)_N = 0 \qquad (2.18a)$$

$$L_1(u_1) + p_{1s} - u_{1NN} = 0 \qquad (2.18b)$$

$$p_{1N} = 0 \qquad (2.18c)$$

$$L_1(h_1 + Eu_1^2/2) - (\sigma^{-1} h_1 + Eu_1^2/2)_{NN} = 0 \qquad (2.18d)$$

where L_r (operator) $= u_r\, \partial/\partial s + v_r\, \partial/\partial N$,

and the subscripts s and N indicates the partial differentiation. These are the well-known Prandtl boundary layer equations. Similarly for the second approximation we get,

$$(r^j u_2)_s + (r^j v_2)_N = -\left[r^j(j\cos\theta/r)Nu_1\right]_s - \left[r^j(K + j\cos\theta/r)\,Nv_1\right]_N \tag{2.19a}$$

$$L_1(u_2) + L_2(u_1) + p_{2s} - u_{2NN} = K(Nu_{1NN} + u_{1N} - Nv_1u_{1N} - u_1v_1) + (j\cos\theta/r)u_{1N} \tag{2.19b}$$

$$p_{2N} = Ku_1^{\,2} \tag{2.19c}$$

$$L_1(h_2 + Eu_1u_2) + L_2(h_1 + Eu_1^{\,2}/2) - (\sigma^{-1}h_2 + Eu_1u_2)_{NN} = KNu_1(h_1 + Eu_1^{\,2}/2)_s + K(\sigma^{-1}h_1 - Eu_1^{\,2}/2)_N + (j\cos\theta/r)(\sigma^{-1}h_1 + Eu_1^{\,2}/2)_N. \tag{2.19d}$$

The above equations are the second order boundary layer equations.

The corresponding inner boundary conditions obtained from (2.8a) and (2.8b) for Prandtl boundary layer equations are

$$u_1(s,\ 0) = 0, \qquad v_1(s,\ 0) = v_w(s) \tag{2.20a}$$

$$h_1(s,\ 0) = h_w(s) \text{ or } h_{1N} = 0 \tag{2.20b}$$

and for the second order boundary layer theory are

$$u_2(s,\ 0) = 0 \qquad v_2(s,\ 0) = 0 \tag{2.21a}$$

$$h_2(s,\ 0) = 0 \quad \text{or} \quad h_{2N}(s,0) = 0. \tag{2.21b}$$

2.4.3 Matching:

The inner expansion is valid in a region of $O(R^{-1/2})$ near the body and in general violates the far upstream conditions. The outer expansion is valid out side this region of $O(R^{-1/2})$ and violates some of the boundary conditions at the

surface (through the loss of highest order derivative of Navier-Stokes equation in Euler limit). Hence, neither expansion has sufficient boundary conditions. This means that these missing boundary conditions must be replaced by something which makes the problem determinate. This is done possibly by what Kaplun (1957) calls the extension theorem, which states that, if we know,

$$\lim_{\zeta} \quad u = 0$$

for some ζ, then these are other limiting process ζ', in the neighbourhood of ζ but not equivalent to it, such that

$$\lim_{\zeta'} \quad u = 0$$

also. Starting with this extension theorem we can easily establish various forms of the matching principles used in practice. First one is generally called the asymptotic matching principle, which states,

The m-term inner expansion of (the p-th term outer expansion) = the p-term outer expansion of (the m-th term inner expansion). (2.22)

The second one states,

Inner limit of the outer expansion = Outer limit of the inner expansion. (2.23)

Van Dyke (1962) applied the first matching principle (2.22) with m = p and m = p-1 and obtained the appropriate matching

conditions. However, we shall prefer to use the second one, i.e., (2.23). We illustrate the procedure by the following example.

Let $\Phi(s, n)$ be a typical variable, whose outer expansion is given by,

$$\Phi(s, n; R) = \Phi_1(s, n) + R^{-1/2}\Phi_2(s, n) + R^{-1}\Phi_3(s, n) + \ldots \quad (2.24)$$

Its inner asymptotic expansion is given by Taylor series (assuming it exists) around n = 0 and using the inner variable this becomes,

$$\Phi(s, n \to 0) = \Phi_1(s, 0) + R^{-1/2}\left[\Phi_2(s, 0) + N\,\Phi_{1n}(s,0)\right] + R^{-1}\left[\Phi_3(s, 0) + N\,\Phi_{2n}(s, 0) + \frac{N^2}{2}\Phi_{1nn}(s, 0)\right] + \ldots$$

On the other hand the outer limit of inner expansion (2.17) is,

$$\Phi(s, N; R) = \phi_1(s, N) + ER^{-1/2}\phi_2(s, N) + R^{-1}\phi_3(s, N) + \ldots, \; N \to \infty. \quad (2.25)$$

Using the matching principle (2.23) we get

$$\begin{aligned} \phi_1(s, N) &= \Phi_1(s, 0) \\ \phi_2(s, N) &= N\,\Phi_{1n}(s, 0) + \Phi_2(s, 0) \qquad \text{as } N \to \infty \\ \phi_3(s, N) &= \frac{N^2}{2}\Phi_{1nn}(s, 0) + N\,\Phi_2(s, 0) + \Phi_3(s,0). \end{aligned} \quad (2.26)$$

Apply these results (2.26), the first one gives the matching condition for the Prandtl boundary layer equations,

$$
\left.\begin{aligned}
u_1(s, N) &= U_1(s, 0) \\
\psi_1(s, N) &= \Psi_1(s, 0) \\
p_1(s, N) &= P_1(s, 0) \\
h_1(s, N) &= H_1(s, 0)
\end{aligned}\right\} \quad \text{as } N \to \infty \tag{2.27}
$$

and the middle one gives for the second order boundary layer

$$
\left.\begin{aligned}
u_2(s, N) &= NU_{1n}(s, 0) + U_2(s, 0) \\
v_2(s, N) &= NV_{1n}(s, 0) + V_2(s, 0) \\
\psi_2(s, N) &= N\Psi_{1n}(s, 0) + \Psi_2(s, 0) \\
p_2(s, N) &= NP_{1n}(s, 0) + P_2(s, 0) \\
h_2(s, N) &= NH_{1n}(s, 0) + H_2(s, 0).
\end{aligned}\right\} \quad \text{as } N \to \infty \tag{2.28}
$$

Various normal derivatives required here are already evaluated in section (2.4.2). Moreover, we also need $\Psi_2(s, 0)$ and $V_2(s, 0)$. It can easily be shown from above that

$$
\Psi_2(s, 0) = \lim_{N \to \infty} (\psi_1 - N\,\psi_{1N}) \tag{2.29a}
$$

$$
V_2(s, 0) = \lim_{N \to \infty} (v_1 - Nv_{1N}). \tag{2.29b}
$$

The quantities on right hand side can be evaluated from the classical boundary layer theory. Physically, the above equations represents the fact that the effect of the classical boundary layer upon the outer flow is that of distribution of sources over the body whose strength is proportional to the slope of the displacement thickness.

2.5 Governing Equations for First and Second Order Boundary Layer Problems:

We now write the final form of the governing equations for the first and second order boundary layer problems.

2.5.1 First Order Boundary Layer Problem:

Integrating the normal momentum equation (2.18c) with respect to N and evaluating the function of integration from the matching condition (2.27c) we get,

$$p_1(s,\ N) \;=\; P_1(s,\ 0). \tag{2.30}$$

This shows that the pressure p_1 is constant across the first order boundary layer. Differentiating (2.30) with respect to s and using (2.14a) we get,

$$p_{1s}(s,\ N) \;=\; P_{1s}(s,\ 0) \;=\; -\,U_1(s,\ 0)U_{1s}(s,\ 0). \tag{2.31}$$

Thus the final form of the first order boundary layer equations is

$$(r^j u_1)_s \;+\; (r^j v_1)_N \;=\; 0 \tag{2.32a}$$

$$L_1(u_1) - u_{1NN} = U_1(s,\ 0)\ U_{1s}(s,\ 0) \tag{2.32b}$$

$$L_1(h_1 + Eu_1{}^2/2) - (\sigma^{-1}h_1 + Eu_1{}^2/2)_{NN} = 0. \tag{2.32c}$$

and the corresponding boundary and matching conditions are

$$u_1(s,\ 0) \;=\; 0 \;=\; v_1(s,\ 0) \tag{2.33a}$$

$$h_1(s,\ 0) \;=\; h_w(s) \quad \text{or} \quad h_{1N}(s,\ 0) \;=\; 0 \tag{2.33b}$$

$$\left.\begin{aligned} u_1(s,\ N) \;&=\; U_1(s,\ 0) \\ h_1(s,\ N) + Eu_1{}^2(s,\ N)/2 \;&=\; S_1(0) \end{aligned}\right|\ \text{as } N \to \infty \tag{2.33c}$$

2.5.2. Second Order Boundary Layer Problem:

Integrating the normal momentum equation (2.19c) with respect to N and using (2.15), (2.16b) and (2.28d) gives,

$$p_2(s,\ N) = KNU_1{}^2(s,\ 0) + K\int_N^\infty [U_1{}^2(s,\ 0) - u_1{}^2(s,N)]\ dN$$
$$+ B'(0)\ \Psi_2(s,\ 0) - U_1(s,\ 0)\ U_2(s,0)\ . \quad (2.34a)$$

Differentiating (2.34a) with respect to s and using the definition of Ψ_2 we get,

$$p_{2s}(s,\ N) = [KNU_1{}^2(s,\ 0) + K\int_N^\infty U_1{}^2(s,\ 0) - u_1{}^2(s,\ N)\ dN]_s$$
$$- r^j\ B'(0)\ V_2(s,\ 0) - [U_1(s,\ 0)U_2(s,0)]_s\ .$$

Substituting the above into (2.19) gives the second order boundary layer equations,

$$(r^ju_2)_s + (r^jv_2)_N = -[r^j(j\ \cos\ \theta/r)Nu_1]_s$$
$$-[r^j(K + j\ \cos\ \theta/r)Nv_1]_N \quad (2.35a)$$

$$L_1(u_2) + L_2(u_1) - u_{2NN} = K[N(u_1u_{1s} - U_1(s,\ 0)U_{1s}(s,0))$$
$$+ u_{1N} - u_1v_1] + [KU_1{}^2(s,0)+K\int_N^\infty (U_1{}^2 - u_1^2)\ dN]_s$$
$$+ (j\ \cos\ \theta/r)\ u_{1N} + r^jB'_1(0)V_2(s,0)+[U_1(s,0)U_2(s,0)]_s$$
$$\quad (2.35b)$$

$$L_1(h_2 + Eu_1u_2) + L_2(h_1 + Eu_1{}^2/2)-(\sigma^{-1}h_2+Eu_1u_2)_{NN}$$
$$= KNu_1(h_1+Eu_1{}^2/2)_s + K(\sigma^{-1}h_1-Eu_1{}^2/2)_N$$
$$+ (j\ \cos\ \theta/r)\ (\sigma^{-1}h_1 + Eu_1{}^2/2)_N\ . \quad (2.35c)$$

The second order boundary conditions are,

$$u_2(s,\ 0) \ = \ 0 \ = \ v_2(s,\ 0) \tag{2.36a}$$

$$h_2(s,\ 0) \ = \ 0 \text{ or } h_{2N}(s,\ 0) \ = \ 0 \tag{2.36b}$$

and the matching conditions as obtained from (2.28) using (2.29) and (2.16) are,

$$\left.\begin{aligned} u_2(s,\ N) \ &= \ N\left[r^j B'_1(0) - K\, U_1(s,\ 0)\right] + U_2(s,\ 0) \\ h_2(s,\ N) \ &+ Eu_1(s,\ N)\, u_2(s,\ N) = \Psi_1(s,\ N)\, S'_1(0) \end{aligned}\right\} \text{as } N \to \infty$$

$$\ldots, \tag{2.36c}$$

CHAPTER 3

FIRST ORDER COMPRESSIBLE BOUNDARY LAYER ON A FLAT PLATE WITH ARBITRARY FLUID PROPERTIES

3.1 Introduction:

Boundary layer flows of a viscous compressible fluids with no pressure gradients depends mainly on the Prandtl number and the viscosity-temperature relationship. Baring, certain specific numerical calculations (see Chapter 1) all the available results assume either the Prandtl number of order unity or the viscosity to be directly proportional to temperature. The aim of the present study is to fill this gap, by studying the flow of a variable thermal properties fluid for three different ranges (i.e., low, high and of order unity) of Prandtl number.

To account for the variation of viscosity with enthalpy we use a power law relation of the type $\mu \propto h^w$. Further, as shown by Peng and Pindroh (1961), the behaviour of a gas at high temperature departs from that of a perfect gas. The gas law, then may be written as

$$p = Z_c \rho R_g T$$

where Z_c is the compressibility factor and R_g is the gas constant. To simplify the problem, we will assume that this gas law can be written as $p = \rho R_g T$ and ρ is approximated by a simple power law relation, for a given pressure $\rho \propto T^a$, 'a' being a constant to be determined by curve fitting. Another

advantage of the law $\rho \propto T^a$ is that the same study can be applied to liquids (with careful interpretation).

Thus in the present study we use the following power law relations for viscosity μ , density ρ and Prandtl number σ with enthalpy h

$$\mu \alpha h^w, \qquad \rho \alpha h^a, \qquad \sigma \alpha h^b.$$

Here, the indices w, a and b are to be determined from the curve fitting to experimental data or by some theoretical considerations. Further, we study the flows for three different ranges of the Prandtl number, i.e., low, high and of order unity. The problem of low Prandtl number is of interest in liquid metal and plasma. For air the Prandtl number at ordinary temperatures is 0.73. Due to high speed flights at high altitudes the temperature increases and pressure falls thus the Prandtl number can become very small (see inset in Fig. 4.6 after Mitchell 1962). The problem of high Prandtl number is of interest in liquid and some vapours. Moreover as said in the first chapter, the study of limiting cases is quite useful even under nonasymptotic conditions. The method used for these limiting cases is that of matched asymptotic expansions. For Prandtl number of order unity we use certain asymptotic method developed by Meksyn (1961).

The present work is not aimed at getting the exact asymptotic solutions (though, in one - low Prandtl number - case the exact results are given) but at finding the analytical dependence of the boundary layer characteristics on various

parameters, in order to obtain a quick overall view of the relative importance of these parameters as well as a guide when looking for correlations.

3.2 Governing Equations:

First order boundary layer equations for a steady flow of a viscous compressible fluid past a flat plate in usual notations are (see, Stewartson 1964, Van Dyke 1961)

$$(\rho u)_s + (\rho v)_n = 0 \tag{3.1a}$$

$$\rho(uu_s + vu_n) = R^{-1}(\mu u_n)_n \tag{3.1b}$$

$$\rho(uh_s + vh_n) = R^{-1}\left[(\mu\sigma^{-1}h_n)_n - \mu u_n^2\right] \tag{3.1c}$$

and the corresponding boundary conditions are

$$n = 0 \qquad u = 0 = v, \qquad h = h_w \text{ or } h_n = 0 \tag{3.2a}$$

$$n \to \infty \qquad u \to 1 \qquad h \to 1. \tag{3.2b}$$

Here the coordinates s along the plate and n normal to it are non-dimensionalised by some reference length L (as there is no characteristic length in the problem, L will not appear explicitly). The velocity components u, v (in s and n directions), enthalpy h, viscosity μ and density ρ are nondimensionalised by their free stream values U_∞, h_∞, μ_∞ and ρ_∞ respectively. The number R is the Reynolds number $U_\infty L\,\rho_\infty/\mu_\infty$.

Introducing the stream function defined by

$$\rho u = \psi_n, \qquad \rho v = -\psi_s.$$

the equation (3.1a) is automatically satisfied while the equations (3.1b) and (3.1c) reduce to total differential equation if we assume ψ and h to be of the form

$$\begin{aligned} \psi(s,\ n) &= (2s/R)^{1/2}\ f(\eta) \\ h(s,\ n) &= h\ (\eta) \end{aligned} \tag{3.3}$$

with $$\eta = (R/2s)^{1/2} \int_0^n \rho \, dn \ .$$

The reduced (total) differential equations are

$$(\rho\mu f'')' + ff'' = 0 \tag{3.4}$$

$$(\sigma^{-1}\rho\mu h')' + fh' + C\rho\mu f''^2 = 0 \ . \tag{3.5}$$

Also the boundary conditions (3.2) become

$$f(0) = 0 = f'(0), \qquad f'(\infty) = 1, \tag{3.6a}$$

$$h(0) = h_w \text{ or } h'(0) = 0,\ h(\infty) = 1 \ . \tag{3.6b}$$

Here dashes denote the differentiation with respect to η.

To account for the variation of ρ, μ and σ with h, we use the power law relations

$$\mu = h^w, \qquad \rho = h^a, \qquad \sigma/\sigma_\infty = h^b \ . \tag{3.7}$$

Eliminating μ, ρ and σ in favour of h from (3.4) and (3.5) we get

$$f''' + f''\left[fh^{-(a+w)} + (a+w)\, h'/h \right] = 0 \tag{3.8}$$

$$h'' + h'\left[\sigma_\infty\, f\, h^{(b-a-w)} - (b-a-w)\, h'/h \right] + C\,\sigma_\infty h^b f''^2 = 0. \tag{3.9}$$

Further introducing the transformation,

$$h = \underset{\sim}{\theta}^{m+1}, \qquad m = \frac{b - a - w}{1 + a + w - b} \tag{3.10}$$

the equations (3.8) and (3.9) reduce to

$$f''' + f''(f\underset{\sim}{\theta}^{e} - e\,\underset{\sim}{\theta}'/\underset{\sim}{\theta}) = 0 \tag{3.11}$$

$$\underset{\sim}{\theta}'' + \sigma_\infty f\underset{\sim}{\theta}^{m}\underset{\sim}{\theta}' + \sigma_\infty \overline{C}\,\underset{\sim}{\theta}^{d-m} f''^{2} = 0 \tag{3.12}$$

where $e = -(a + w)(1 + m)$, $d = b(1 + m)$ and $\overline{C} = C/(1+m)$. The boundary conditions for the velocity profile are same as (3.6a) and those for temperature profile are

$$\underset{\sim}{\theta}(0) = \theta_w = h_w^{1/(1+m)} \text{ or } \underset{\sim}{\theta}'(0) = 0, \quad \underset{\sim}{\theta}(\infty) = 1. \tag{3.13}$$

3.3 Solution for Low Prandtl Number:

Edward and Tellep (1961) have analysed the heat transfer in low Prandtl number fluids with variable thermal properties by ignoring the viscous dissipation and pressure gradient. To account for the variation of thermal properties with temperature these authors have used for the term $\rho\mu/\sigma$ in the energy equation a power low dependence of the type T^m (in the notations of Edward and Tellep). They have solved the problem for the range of m between zero and unity ($0 \leq m \leq 1$). However, for a perfect gas and most of the fluids m is usually between zero and minus one ($-1 \geq m \geq 0$). It is for this range that we here study the heat transfer problem with full account of dissipation. In terminology of the method of matched asymptotic expansions (as employed here), Edward and Tellep have solved the

so called the zeroth order outer equation; their solution is however, uniformly valid up to the order $\sigma^{1/2}$ for the heat transfer case. To get the higher order terms it is essential to formulate a different inner limit of the equations. In the present work we have formulated and solved the equations for the two regions with full account of dissipation. Also the study of an insulated wall in the flow of a fluid with arbitrary properties is included.

We now solve the equations (3.11) and (3.12) under the boundary conditions (3.6a) and (3.13) at low σ_∞. Following Lagerstrom (1964) we proceed as follows.

3.3.1 Inner limit: The inner limit is defined as $\sigma_\infty \to 0$ with η fixed, and we write

$$f(\eta, \sigma_\infty) = \sum_{n=0} f_n(\eta)\, \sigma_\infty^{n/2}, \tag{3.14a}$$

$$\underset{\sim}{\theta}(\eta, \sigma_\infty) = \sum_{n=0} \theta_n(\eta)\, \sigma_\infty^{n/2}. \tag{3.14b}$$

Substituting the above series (3.14) in equation (3.11) and (3.12) and collecting the coefficient of the like power of σ_∞, the momentum equation (3.11) gives,

$$f_o''' + f_o''(f_o\theta_o^e - e\theta_o'/\theta_o) = 0 \tag{3.15a}$$

$$f_1''' + f_1''(f_o\theta_o^e - e\theta_o'/\theta_o) + f_o''(f_1\theta_o^e + ef_o\theta_o^{e-1}\theta_1 - e\theta_1'/\theta_o + e\theta_o'\theta_1/\theta_o^2) = 0, \tag{3.15b}$$

and the energy equation (3.12) gives

$$\theta_0'' = 0 \qquad (3.16a)$$

$$\theta_1'' = 0 \qquad (3.16b)$$

$$\theta_2'' = -f_0\theta_0^m - \bar{C}\,\theta_0^{d-m}\,f_0''^2 . \qquad (3.16c)$$

The solutions to above three equations (3.16) are

$$\theta_0 = a_0\eta + b_0 \qquad (3.17a)$$

$$\theta_1 = a_1\eta + b_1 \qquad (3.17b)$$

$$\theta_2 = -a_0\int_0^{\eta}(\eta - x)(a_0x + b_0)^m\, f_0(x)\, dx \qquad (3.17c)$$

$$- \bar{C}\int_0^{\eta}(\eta - x)(a_0x + b_0)^{d-m}\, f_0''^2(x)dx + a_2\eta + b_2 .$$

Here a's and b's are constant of integration either of them to be determined from inner boundary conditions.

All these solutions (3.17) are singular for large η and do not satisfy the boundary conditions at infinity. This singularity is rather similar to one encountered in improving Stokes solution for low Reynolds number flow.

3.3.2 Outer limit: We now need a different outer limit. From an order of magnitude analysis we introduce the outer variables

$$\zeta = \sigma_\infty^{1/2}\eta, \qquad F = \sigma_\infty^{1/2} f \qquad \phi(\zeta) = \theta(\eta) \qquad (3.18)$$

and study the limit $\sigma_\infty \rightarrow 0$ with ζ, F and ϕ fixed. With this outer limit the equations (3.12) and (3.13) become

$$F F_{\zeta\zeta} \phi^e + \sigma_\infty^{-1/2} (F_{\zeta\zeta\zeta} - eF_{\zeta\zeta} \phi_\zeta / \phi) = 0 \tag{3.19a}$$

$$\phi_{\zeta\zeta} + F\phi^m \phi_\zeta + \bar{C} \sigma_\infty \phi^{d-m} F^2_{\zeta\zeta} = 0. \tag{3.19b}$$

It can be shown easily that the solution of the momentum equation (3.19a) which satisfies the boundary condition at infinity is

$$F = \zeta - \sigma_\infty^{-1/2} \alpha + O(\sigma^{-\infty}). \tag{3.20}$$

Where $\sigma^{-\infty}$ denotes the exponentially small terms in the limit $\sigma \to \infty$. Now with the help of (3.20), the outer energy equation (3.19b) may be written as

$$\phi_{\zeta\zeta} + (\zeta - \sigma_\infty^{-1/2}\alpha) \phi^m \phi_\zeta = O(\sigma^{-\infty}). \tag{3.21}$$

The outer boundary condition is $\phi(\infty) = 1$. The nonlinear outer equation (3.21) is correct to all orders in σ_∞, i.e., the error is exponentially small. This is due to the fact that there is a small momentum layer inside the thick thermal layer and all that the momentum layer does far away is to displace the stream lines from their inviscid position by an amount α. The outer equations of all orders can be obtained from equation (3.21). When $m = 0$, the solution to this equation (3.21) which satisfies the boundary condition at infinity is

$$\phi = A + (1 - A) \operatorname{erf}\left[(\zeta - \sigma_\infty^{-1/2}\alpha)/\sqrt{2}\right]$$

where erf (x) is the well-known error function defined by

$$\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$$

and $\quad A = \Sigma A_n \sigma_\infty^{n/2}.$

It appears that for a non zero m, in general, equation (3.21) has to be solved numerically. However, it is interesting to see that for an insulated wall the solution of zeroth order outer equation $\phi_o = 1$ matches with the corresponding inner solution; so that it is not necessary to solve the equation (3.21) numerically. Therefore, we now study the outer equation (3.21) separately for the cases of the insulated wall and heat transfer.

3.3.4: Insulated wall case: Before solving the nonlinear outer equation (3.21), let us recall that for an insulated wall $\underset{\sim}{\theta}'(0) = 0$, and the inner equations (3.17) gives $a_o = a_1 = .. = 0$. Now the outer limit of the inner solution for large η may be written an

$$\underset{\sim}{\theta}(\eta \to \infty) = b_o + \sigma_\infty^{1/2} \left(b_1 + \bar{C}\, b_o^{d-m} \int_0^\infty f_o''^2(x)\, dx\right) + .. \quad (3.22)$$

We now proceed to solve the outer equation (3.21). Let us assume

$$\phi(\zeta, \sigma_\infty) = \sum_{n=0} \phi_n(\zeta)\, \sigma_\infty^{n/2}. \quad (3.23)$$

Substituting this expansion (3.23) in equation (3.21) and collecting the coefficients of the like powers of σ_∞, we get

$$\phi_{o\zeta\zeta} + \zeta \phi_o^m \phi_{o\zeta} = 0 \quad (3.24a)$$

$$\phi_{1\zeta\zeta} + \zeta \phi_o^m (\phi_{1\zeta} + m\phi_1 \phi_{o\zeta}/\phi_o) = \alpha \phi_o^m \phi_{o\zeta} \quad (3.24b)$$

$$\phi_{2\zeta\zeta} + \zeta\phi_0^m\,(\phi_{2\zeta} + m\,\phi_2\,\phi_{0\zeta}\,/\phi_0) = \phi_0^m(\alpha\phi_{1\zeta} + \alpha m\phi_1\phi_{0\zeta}\,/\phi_0 - \zeta m\,\phi_1\phi_{1\zeta}/\phi_0 - (m/2)(m-1)\zeta\phi_1^2\,\phi_{0\zeta}\,/\phi_0^2). \tag{3.24c}$$

The boundary conditions at infinity are

$$\phi_0\,(\infty) = 1, \quad \phi_1(\infty) = 0, \quad \phi_2(\infty) = 0, \;\ldots\ldots \tag{3.25}$$

The solution of (3.24a) which satisfies the boundary condition $\phi_0(\infty) = 1$ and matches with (3.22) upto the zeroth order is

$$\phi_0(\zeta) = 1 = b_0\,. \tag{3.26}$$

The first order outer equation (3.24b) now becomes

$$\phi_{1\zeta\zeta} + \zeta\phi_{1\zeta} = 0$$

and its solution with boundary condition $\phi_1(\infty) = 0$ is

$$\phi_1 = A\,\mathrm{erfc}\,(\zeta/\sqrt{2}) \tag{3.27}$$

where erfc (x) is the complementary error function defined by

$$\mathrm{erfc}\,(x) = 1-(2/\sqrt{\pi})\int_0^x e^{-t^2}\,dt.$$

For small ζ the inner expansion of (3.27) is

$$\phi_1(\zeta \to 0) = A\,(1-(2/\,\pi)^{1/2}\zeta + \ldots\,). \tag{3.28}$$

Matching (3.28) with the first order terms in (3.22); we get

$$b_1 = A = \overline{C}\,(\pi/2)^{1/2}\int_0^\infty f_0''^{\,2}(x)\,dx. \tag{3.29}$$

Now with the help of the relation (3.26) and the solution (3.9) of the outer momentum equation (3.15a) reduces to the Blasius

equation; we denote the solution $f_0(\eta)$ by $B(\eta)$.

The adiabatic wall temperature is

$$h_r = 1 + C\,(\pi\sigma_\infty/2)^{1/2} \int_0^\infty B''^2(x)\,dx, \qquad (3.30a)$$

and the recovery factor is

$$\begin{aligned} r_f &= 2(h_r - 1)/C \\ &= \sqrt{(2\pi\sigma_\infty)} \int_0^\infty B''^2(x)\,dx = 0.9255\ \sigma_\infty^{1/2}. \end{aligned} \qquad (3.30b)$$

This last result is independent of thermal properties of the fluid.

3.3.5. Heat Transfer Case: In this case the nonlinear outer equation (3.21) can be solved exactly. It is convenient to change the following independent variable,

$$z = \zeta - \sigma_\infty^{1/2}\,\alpha.$$

The equation (3.21) reduces to

$$\phi_{zz} + z\phi^m\phi_z = 0 \qquad (3.31)$$

and the corresponding boundary condition is $\phi(\infty) = 1$.

To solve the equation (3.31) (which is still correct to all order in σ_∞) numerically, it is first to be transformed into a marching problem by setting

$$\phi(z) = l\,T(t), \quad z = q_1 t \qquad (3.32a)$$

where l and q_1 are constants. Substituting (3.32a) in (3.31), we get,

$$T_{tt} + q_1^2\, l^m\, t\, T^m\, T_t = 0. \tag{3.32b}$$

If $\quad q_1^2\, l^m = 1$

the equation (3.32b) reduces to

$$T_{tt} + t\, T^m\, T_t = 0. \tag{3.32c}$$

This equation (3.32c) is similar to equation (3.31) but the boundary condition at infinit- is changed, which is $T(\infty) = 1/l$. This makes possible to generate the family of solutions. Now for a given 'm' assume a value of $T(0) = 1$ and set the values for $T_t(0)$. For each value of $T_t(0)$ the equation (3.32c) is solved numerically by Runge-Kutta method (on IBM 7044 computer at Indian Institute of Technology Kanpur) to get $T(\infty)$. Knowing $T(\infty)$ the constants l and q_1 are determined by

$$l = (T(\infty))^{-1}$$

$$\text{and} \quad q_1 = (T(\infty))^{m/2}. \tag{3.32d}$$

The equation (3.32c) and its solution may now be transformed to original variables $\emptyset$ and z with the aid of (3.32b, d). Solutions are obtained for a range of values of m between zero and unity $(0 < m \leq 1)$. The tabulation of all the results of the numerical computation is rather difficult. Instead we shall be displaying in Fig. 3.3 the initial values of each of the solutions. Only some of the solutions for outer temperature profile are shown in Figs. 3.1 and 3.2.

The results of numerical solution, necessary for matching are plotted in Fig. 3.3, which displays $\phi_z(0)$ $V_s\phi(0)$ for various values of m. To affect the matching, we write formally the inner expansion of the (numerical) outer solution as

$$\phi(\zeta \to 0) = \phi(0) + \sigma_\infty^{1/2} (\eta - \alpha)\, \phi_z(0) + \ldots . \tag{3.33}$$

Here $\phi(0)$ and $\phi_z(0)$ are constants independent of z but are functions of σ_∞; say

$$\phi(0) = \sum_{n=0} A_n \sigma_\infty^{n/2}, \quad \phi_z(0) = \sum_{n=0} C_n \sigma_\infty^{n/2}. \tag{3.34}$$

Substituting (3.34) in (3.33), we get the inner expansion of the outer solution as $\zeta \to 0$

$$\phi(\zeta \to 0) = A_o + \sigma_\infty^{1/2} (A_1 - \alpha C_o + \eta C_o) + \sigma_\infty (A_2 - \alpha C_1 + \eta C_1) + O(\sigma_\infty^{3/2}). \tag{3.35}$$

First matching the zeroth order inner solution (3.17a) with the zeroth order outer solution (3.35) we get $a_o = 0$. Next the outer expansion of the inner solution (3.17) may be written as (by the inner boundary condition $b_o = \theta_w$, $b_1 = b_2 = 0$)

$$\theta(\eta \to \infty) = \theta_w + a_1 \eta \sigma_\infty^{1/2} + \sigma_\infty \Big\{ \eta (a_2 + \bar{C} \theta_w^{d-m} \int_o^\infty f_o''^2(x)\, dx) - \bar{C}\, \theta_w^{d-m} \int_o^\infty x\, f_o''^2 (x)\, dx \Big\} + O(\sigma_\infty^{3/2}). \tag{3.36}$$

Matching (3.35) and (3.36) we get,

$$a_o = 0, \qquad A_o = \theta_w$$

$$a_1 = C_o \qquad A_1 = C_o \alpha$$

$$a_2 = C_1 - \overline{C}\,\theta_w^{d-m} \int_0^\infty f''^2(x)\,dx$$

$$A_2 = C_1\alpha - \overline{C}\,\theta_w^{d-m} \int_0^\infty x\,f''^2(x)\,dx\,.$$

It can be shown easily that the matching result (3.37) together with the solution of the outer equation (3.20) and the transformation

$$f_0(\eta) = B(\bar\eta)\,\theta_w^{-n/2}\,, \qquad \eta = \bar\eta\,\theta_w^{-n/2} \tag{3.38}$$

reduce the zeroth order inner momentum equation (3.15a) to the well-known Blasins equation with the solution $B(\bar\eta)$.

The heat transfer rate at the wall is

$$\underset{\sim}{\theta}'(0) = \sigma_\infty^{1/2}\,C_0 + \sigma_\infty\Big[C_1 + \overline{C}\theta_0^{d-m+(n/2)} \int_0^\infty B''^2(x)dx\Big] + \ldots \tag{3.39}$$

Introducing the recovery temperature (3.30) we get

$$h'(0) = \sigma_\infty^{1/2}\Big[(m+1)\,h_w^{\frac{m}{m+1}}\,\phi_z(0) + \sqrt{(2/\pi)}\,(h_r-1)\,h_w^{\frac{m}{2(m+1)}+\frac{b}{2}}\Big] + O(\sigma_\infty)\,. \tag{3.40}$$

Let us rewrite the above relation as

$$h'(0) = W_1 + \sqrt{\frac{2}{\pi}}\,(h_r-1)\,h_w^{\frac{m}{2(1+m)}+\frac{b}{2}}\,\sigma_\infty^{1/2} \tag{3.41}$$

where

$$W_1 = (m+1)\,h_w^{\frac{m}{m+1}}\,\phi_z(0)\,\sigma_\infty^{1/2}\,. \tag{3.42}$$

The quantity W_1 may be recognized as the heat transfer rate if viscous dissipation is ignored. This factor $W_1/\sigma_\infty^{1/2}$ is plotted against h_w in Fig. 3.6 with m as a parameter. For a given value of h_w, we first determine $\phi(0) = h_w^{1/(1+m)}$. Next from Fig. 3.3 we obtain $\phi_z(0)$ for a given $\phi(0)$ and m. This procedure determines W_1. Knowing the recovery temperature, heat transfer can be determined from (3.43).

For the case of perfect gas (a = -1) and viscosity proportional to temperature (w = 1) and constant Prandtl number (b = 0) this becomes,

$$W_1 = \phi_z(0)\, \sigma^{1/2} = (1 - h_w)\sqrt{(2\sigma/\pi)}$$

and

$$h'(0) = (h_r - h_w)\sqrt{(2\sigma/\pi)} \quad .$$

This last result is same as given in Stewartson (1964).

The coefficient of skin friction C_f is defined as

$$C_f = (\mu u_n)_{n=0} = \left(\frac{2}{RS}\right)^{1/2} (\rho\mu f'')_{\eta=0} \; . \qquad (3.43)$$

With the help of (3.38), (3.7) and (3.10) the above result becomes

$$C_f \sqrt{R_x} = \sqrt{2}\, h_w^{(w+a)/2}\, B''(0) + O(\sigma_\infty^{-1/2})$$
$$= 0.6614\; h_w^{(w+a)/2} + O(\sigma_\infty^{-1/2}). \qquad (3.44)$$

The uniformly valid solution to enthalpy profile is

$$h(\eta) = h_w + C\sigma_\infty \left[\int_0^\eta x\, f_o''^2(x)\, dx + \eta \int_\eta^\infty f_o''^2(x)\, dx\right] \qquad (3.45)$$
$$+ (1+m)\, h_w^{m/(1+m)} \left[\phi\left\{\sigma_\infty^{1/2}.(\eta - \alpha)\right\} - \phi\,(-\sigma_\infty^{1/2}\alpha)\right].$$

The above solution shows that the temperature increases like η^{-1} as $\eta \to 0$. Thus it not only fails to satisfy the boundary condition at the wall, but is singular there.

3.4.2 Inner Limit:

We need now a different inner limit process. From an order of magnitude analysis we introduce the following inner variables

$$
\begin{aligned}
y &= \eta/\epsilon \\
\chi &= \underset{\sim}{\theta}\, \sigma_\infty^{-1/(5m-4e+3-3d)} \qquad (3.49) \\
F &= f\, \sigma_\infty^{(4m-3e-2d+1)/(5m-4e-3d+3)}
\end{aligned}
$$

with $\epsilon = \sigma_\infty^{-(2m-e-d+1)/(5m-4e-3d+3)}$

and study the limit $\sigma_\infty \to \infty$ with y, F and χ fixed. Assuming

$$
\begin{aligned}
F &= F_0 + \epsilon F_1 + \ldots \\
\chi &= \chi_0 + \epsilon \chi_1 + \ldots
\end{aligned} \qquad (3.50)
$$

the zeroth order inner equations are

$$F_{0yyy} - e\, F_{0yy}\, \chi_{0y}/\chi_0 = 0 \qquad (3.51a)$$

$$\chi_{0yy} + F_0\, \chi_0^m\, \chi_{0y} + \bar{c}\, \chi_0^{d-m}\, F_{0yy}^2 = 0. \qquad (3.51b)$$

These equations (3.51a) and (3.51b) are coupled and nonlinear with the outer boundary conditions to be determined by matching. It turns out that the outer boundary condition for χ_0 is singular at large y. Thus the equations (3.51) and the accompanying boundary conditions leads to very difficult boundary value problem. As our purpose is to find the analytical

dependence, it is instructive to carry out just the first step in Weyl's iteration scheme. To accomplish this, we first (formally) integrate the equations (3.52a) and (3.52b) to yield

$$F_o = a_s \int_0^y (y - x) \chi_o^e(x)\, dx \tag{3.52a}$$

$$\frac{\chi_o^{2m-d+1}}{2m-d+1} = A_o + A_1 \int_0^y \chi_o^{2m-d}(x)\, e^{-\bar{F}(x)}\, dx - \bar{C} \int_0^y \chi_o^{2m-d}(x)\, e^{-\bar{F}(x)} \int_0^x \chi_o^{d-m}(t)\, F^2_{o_{tt}}(t)\, e^{\bar{F}(t)}\, dt\, dx \tag{3.52b}$$

where $\bar{F}(y) = \int_0^y F_o \chi_o^m \, dy$.

Upto this stage the analysis is exact. To study the first step in Weyl's iteration scheme, we use only one term in the expansion of χ_o near the wall, i.e. χ_w. It is observed in view of the fact, that the Prandtl number is high, the thermal boundary layer is very thin and is contained in the viscous boundary layer which itself has small thickness, the above approximation should be very good in the inner region of thermal boundary layer.

For this approximation

$$F_o = \frac{a_s \chi_w^e}{3!} y^3 , \qquad \bar{F} = \frac{a_s}{3!} \chi_w^{m+e} y^3$$

$$\frac{\chi_o^{2m-d+1} - \chi_w^{2m-d+1}}{2m-d+1} = \frac{2A_1}{(36a_s)^{1/3}} \chi_w^{\frac{5m-e-3d}{3}} \gamma(\tfrac{1}{3}, \bar{F}) - \frac{2C}{1+m} (a_s^4/6)^{1/3} \chi_w^{(4m+e/3)} \underline{N}_{1/3}(1/3, \bar{F}) \tag{3.53}$$

where

$$\gamma(\alpha, x) = \int_0^x e^{-t}\, t^{\alpha-1}\, dt$$

is the well-known incomplete gamma function and

$$N_m(\alpha, x) = \int_0^x t^m\, N(\alpha, t)\, dt$$

and $$N(\alpha, x) = \alpha \int_0^1 e^{-xt} (1-t)^{\alpha-1}\, dt$$

are certain related gamma functions studied in Appendix. A.

Further, the outer expansion of this inner solution is

$$\frac{\chi_o^{2m-d+1} - \chi_w^{2m-d+1}}{2m-d+1} = \frac{2ca_s \chi_w^e}{(1+m)} + \frac{A_1}{(36a_s)^{1/3}} \chi_w^{\frac{5m-e-3d}{3}} \Gamma(\tfrac{1}{3})$$

$$- \frac{12c}{m+1} \left(\frac{a_s}{6} \right)^{4/3} \chi_w^{\frac{4m+e}{3}} N_{-1/3}(1/3, \infty).$$

Now the inner expansion of the outer solution (3.48) in inner variables (3.50) may be written as

$$\frac{\chi_o^{2m-d+1}}{2m-d+1} = \frac{2ca_s \chi_w^e}{(1+m)} + \left[\frac{1}{2m-d+1} + \frac{cJ(0)}{m+1} \right] \sigma_\infty^{-\frac{1+2m-d}{5m-4e-3d+3}} + \ldots \qquad (3.55)$$

where

$$J(\eta) = \int_\eta^\infty f_o''/f_o - 2a_{s1}/\eta^2\, d\eta$$

is regular near the wall and $a_{s1} = a_s \chi_w^e$.

3.4.3 Results:

Various unknown constants will now be determined by matching the inner and outer solutions in the overlap domain.

For the case of an insulated wall $\chi'(0) = 0$, requires $A_1 = 0$ and the matching condition (2.24) gives the wall recovery factor

$$r_f = \frac{2}{C}\left[\frac{2C(2m-d+1)}{1+m}(a_s^4/6)^{1/3} N_{-1/3}(1/3,\infty)\,\sigma_\infty^{1/3}\right]^{\frac{3(m+1)}{5m-4e-3d+3}}$$

$$= \frac{2}{C}\left[2C(1-a-w)(a_s^4/6)^{1/3} N_{-1/3}(1/3,\infty)\,\sigma_\infty^{1/3}\right]^{3/(2w+2a-b+3)}$$

$$= (\bar{r}_f/C)(Ca_s^{4/3})^{3/(2+2a-b-3)}. \qquad \ldots \qquad (3.56)$$

When wall temperature is prescribed to a constant value, the matching principle (2.24) determines

$$A_1 = (\chi_r^{2m-d+1} - \chi_w^{2m-d+1})\,\frac{(36a_s)^{1/3}}{(2m-d+1)\,\Gamma(1/3)}\,\chi_w^{(e-5m+3d)/3}$$

$$= \left(\frac{\partial\chi}{\partial y}\right)_{=0} = \left(\frac{\partial\theta}{\partial\eta}\right)\sigma_\infty^{-\frac{2m-e-d+2}{5m-4e-3d+3}}.$$

The heat transfer rate at the wall is

$$h'(0) = \left[h_r(h_r/h_w)^{\frac{m-d}{m+1}} - h_w\right]\frac{(36a_s)^{1/3}}{\alpha\,\Gamma(1/3)}\,h_w^{\frac{e+m}{3(m+1)}}\,\sigma_\infty^{1/3} + \ldots \quad (3.57a)$$

$$= \left[h_r(h_r/h_w)^{-a-w} - h_w\right]\frac{(36a_s)^{1/3}}{(1-a-w)\,\Gamma(1/3)}\,h_w^{\frac{1}{3}(b-2a-2w)}\,\sigma_\infty^{1/3} + \ldots$$

To study the above result (3.57a) in detail let us write

$$h'(0) = W_h\,a_s^{1/3} - (1-h_r^{1-a-w})\,\frac{(36a_s)^{1/3}\,\sigma_\infty^{1/3}\,h_w^{\frac{1}{3}(b-5a-5w)}}{(1-a-w)\,\Gamma(1/3)}$$

$$\ldots (3.57)$$

where

$$W_h(m,h) = (h_w^{-a-w} - h_w)\,\frac{(36)^{1/3}}{(1-a-w)\,\Gamma(1/3)}\,h_w^{\frac{1}{3}(b-2a-2w)}\,\sigma_\infty^{1/3} \quad (3.58)$$

is the contribution to heat transfer when dissipation is neglected. Knowing $W_h(m, h_w)$ and the recovery temperature actual heat transfer may be obtained from (3.57).

Now for a perfect gas ($a = -1$) with viscosity proportional to temperature ($w = 1$) and constant Prandtl number ($b = 0$), the result (3.56) for the recovery factor reduces to

$$r_f = 1.922\,\sigma^{1/3} \tag{3.59}$$

and (3.56) for the heat transfer becomes

$$h'(0) = 1.6227\,(h_w - h_r)\,\sigma^{1/3}. \tag{3.59b}$$

There results (3.59a) and (3.59b) are same as given in Stewartson (1964).

3.5 <u>Solution for Prandtl Number of Order Unity</u>:

In the previous sections (3.3) and (3.4) it is seen that the low and high Prandtl number problems are of singular perturbation type. However, when Prandtl number is of order unity, the thickness of momentum boundary layer is of the same order as that of thermal boundary layer, so, in this case the problem is expected to be a regular perturbation type. Now to solve

$$f''' + f''\,(fh^e - e\,h'/h) = 0 \tag{3.60a}$$

and

$$h'' + h'\,(\sigma_\infty\, fh^q - qh'/h) + C\,\sigma_\infty h^b\, f''^2 = 0 \tag{3.60b}$$

where $e = -(a+w)$ and $q = b-a-w$, with the boundary conditions (3.6a, b), when Prandtl number is of order unity we utilize a certain asymptotic method developed and used extensively, on many problems, by Meksyn (1961).

The equations (3.60a) and (3.60b) may formally be integrated to give

$$f' = \bar{a} \int_0^{\eta} (h/h_w)^e \exp(-F_n)\, d\eta, \qquad (3.61a)$$

$$h = A_0 + A_1 \int_0^{\eta} (h/h_w)^q \exp(+\sigma_\infty F_m)\, d\eta - C\sigma_\infty \bar{a}^2 \Phi(\eta) \qquad (3.62b)$$

where

$$\Phi(\eta) = h_w^b \int_0^{\eta} \left(\frac{h}{h_w}\right)^q \exp(\mp\sigma_\infty F_m)\, d\eta_2 \int_0^{\eta_2} \left(\frac{h}{h_w}\right)^{2e-q+d} \exp(\sigma_\infty F_m - 2F_n)\, d\eta_1$$

and $\quad F_m(\eta) = \int_0^{\eta} f h^m \, d\eta .$

In writing down the equation (3.61a) we have used the boundary condition at the wall and the condition at infinity is to be satisfied by a proper choice of $\bar{a}$. The constant of integration A_o and A_1 in (3.62b) are yet to be determined.

For an insulated wall, the condition $h'(0) = 0$ leads to $A_1 = 0$ and condition at infinity gives

$$h = 1 + C\sigma_\infty \bar{a}^2 \left[\Phi(\infty) - \Phi(\eta) \right].$$

The wall recovery enthalpy is

$$h_r = 1 + C\sigma_\infty \bar{a}^2 \Phi(\infty) . \qquad (3.63)$$

When wall temperature is prescribed $h(0)=h_w=A_o$ and the condition at infinity determines A_1. Hence the solution to enthalpy profile is

$$h = h_w + (h_r - h_w) \int_0^{\eta} \exp(-\sigma_\infty F_q) d\eta \Big/ \int_0^{\infty} \exp(-\sigma_\infty F_q) d\eta - C\sigma_\infty \bar{a}^2 \Phi(\eta)$$

Heat transfer rate at the wall is

$$h'(0) = (h_r - h_w) \Big/ \int_0^\infty \exp\,(-\sigma_\infty F_q)\, d\eta . \tag{3.64}$$

As our aim is only to obtain an expression for the analytical dependence of boundary layer characteristics on various parameter it is enough to use only one term in the expansion of f and h near the wall, viz

$$f = \frac{a}{2!}\,\eta^2 + \ldots , \qquad h = h_o + \ldots . \tag{3.65}$$

Here $h_o = h_r$ for an insulated wall and $h_o = h_w$ when the wall enthalpy is prescribed.

Now to obtain the value of 'a' we make use (3.61a), (3.65) and boundary condition at infinity gives

$$1 = \bar{a} \int_0^\infty \exp\left(- \frac{\bar{a}}{6}\, h_o^e\, \eta^3 \right) d\eta$$

or
$$\bar{a} = \left[\frac{9h_o^e}{2\ \ ^3(1/3)} \right]^{1/2} \equiv a_s\, h_o^{e/2} . \tag{3.66}$$

The coefficient of skin friction C_f is given by

$$C_f \sqrt{R_s} = \sqrt{2}\, h_o^{-(a+w)}\, \bar{a}$$
$$= 0.684\, h_o^{(a+w)/2}$$

where R_s is local Reynolds number.

The integral Φ is evaluated by using the following relation proved in Appendix A

$$\int_0^\infty Z^m \exp\,(-PZ) \int_0^Z t^n \exp\,(Pt - at)\, dt\, dz \tag{3.67}$$

$$= \frac{(m+n+2)}{n+1}\ {}_2F_1\left(n+1,\ m+n+2,\ n+2,\ \frac{P-a}{P}\right) P^{-(m+n+2)}$$

where ${}_2F_1$ (a, b, c; x) is the hypergeometric function. Now the expression (3.63) for the recovery enthalpy becomes

$$h_r = 1 + \frac{3\ C\,\sigma_\infty^{1/3}}{\Gamma^2(1/3)}\,\Gamma(2/3)\ {}_2F_1(1/3,\ 2/3,\ 4/3,\ \frac{\sigma_\infty - 2h_r^b}{\sigma_\infty})\,h_r^{b/3}.$$

The recovery factor is

$$r_f = 1.29\ {}_2F_1\,(1/3,\ 2/3,\ 4/3,\ \frac{\sigma_\infty - 2h_r^b}{\sigma_\infty})\,h_r^{b/3}\,\sigma_\infty^{1/3} + ..\qquad (3.69)$$

when b = 0 and σ_∞ = 1, this gives $r_f = 1.11$

For heat transfer case the equation (9.64) becomes

$$h'(0) = \frac{(36a_s)^{1/3}}{2\,\Gamma(1/3)}\,(h_r - h_w)\,h_w^{(b-2a-2w)/3}\,\sigma_\infty^{1/3}.\qquad (3.70)$$

Here again to separate the effect of Mach number from that of wall temperature we rewrite (3.70) a

$$h'(0) = W_0(b,a,w;\,h_w)\,a_s^{1/3} - (1-h_r)\,\frac{(36a_s)^{1/3}}{2\,\Gamma(1/3)}\times$$
$$\times\, h_w^{(b-2a-2w)/3}\,\sigma_\infty^{1/3}\qquad (3.71)$$

where

$$W_0 = \frac{(36)^{1/3}}{2\,\Gamma(1/3)}\,(1-h_w)\,h_w^{(b-2a-2w)/3}\,\sigma_\infty^{1/3}\qquad (3.72)$$

is the contribution to heat transfer from wall temperature.

Lastly, apart from the approximate analysis described in this section (3.5), the energy equation (3.60b) can be integrated exactly when σ_∞= 1 and b = 0. To accomplish this, we eliminate the factor fh^e - eh/h from equations (3.60a) and (3.60b) to obtain

$$h'' - h' f''' + C f''^2 = 0 . \qquad (3.73)$$

The solution to this equation is

$$h = l_1 + l_2 f' - \frac{C}{2} f'^2 \qquad (3.74)$$

where l_1 and l_2 are constants of integration The boundary condition $h(\infty) = 1$ gives

$$l_1 + l_2 = 1 + C/2 .$$

For an insulated wall $l_2 = 0$ and the enthalpy profile is

$$h = 1 + \frac{C}{2} (1 - f'^2). \qquad (3.75)$$

The wall enthalpy is $h_r = 1 + C/2$ and the recovery factor is

$$r_f = 1. \qquad (3.76)$$

For heat transfer case $l_1 = h_w$ and $l_2 = h_r - h_w$ and the enthalpy profile is

$$h = h_w + (h_r - h_w) f' - Cf'^2/2. \qquad (3.77)$$

The heat transfer at the wall is

$$h'(0) = (h_r - h_w) f''(0) \qquad (3.78)$$

and the Reynolds analogy factor is

$$RA = \frac{h'(0)}{(h_r - h_w) f''(0)} = 1. \qquad (3.79)$$

These results (3.76) and (3.79) are same as classical Crocco's integral. Thus at $\sigma = 1$, the recovery factor and the Reynolds analogy factors are independent of thermal properties. The results (3.68) and (3.72) for the perfect gas ($a = -1$) and constant Prandtl number ($b = 0$) reduces to those of Vasantha and Narasimha (1969).

3.6 Discussion:

The main results for low, high and order unity ranges of Prandtl number are contained respectively in (3.30), (3.56) and (3.69) for the recovery factor and in (3.41), (3.57) and (3.70) for the heat transfer. The main interest centers on the leading terms, as these indicate the extent to which the results are useful. For the case of a perfect gas (a = -1) with viscosity proportional to temperature (w = 1) and constant Prandtl number (b = 0) our results reduce to those given by various authors. It is interesting to note that the result (3.30) for the recovery factor at low σ_∞ upto the order $\sigma_\infty^{1/2}$ is independent of the thermal properties of the fluid. It seems that the precise dependence of ρ, μ and σ on the enthalpy h is not crucial when the Prandtl number is low. This is plausible, because the plate is thermally insulated and the fluid is highly conducting (low σ_∞); therefore the variation of temperature across the boundary is small. Further, the result (3.69) for the recovery factor when the Prandtl number is of order unity, is independent of w and a and depends on b only. Thus, here it also appears that precise viscosity and density laws are not crucial. The numerical solution of Crocco (1941) for Prandtl number of order unity indicate that the corresponding numerical values using different viscosity laws are close together. Further, Kuerti's (1950) review makes it clear that when wall is insulated and the Prandtl

number held fixed near unity, the temperature variation in the boundary layer for various viscosity laws is approximately the same function of velocity. However, the result (3.56) for the recovery factor when Prandtl number is high depends strongly on w, a, b (the form of property relations of the fluid) as well as on Mach number. The effect of the thermal properties on $\bar{r}_f$ (proportional to recovery factor) is shown in Fig. 3.5. It is seen that $\bar{r}_f$ increases as m increases.

From Fig. 3.4 which displays the recovery factor for above three cases it is seen that low Prandtl number analysis gives 0.9255 (the exact result of Pohlhansen (1921) is unity) while at $\sigma_\infty = 15$ gives r = 3.59 (the exact result of Pohlhansen is 3.54). It is fortuitous that quite useful estimates of recovery factor can be obtained even for $\sigma_\infty = 25$ with in 8 per cent of error. The expression (3.69) for the Prandtl number of order unity is close to the result of Pohlhansen at $\sigma_\infty = 0.7$ and 15. The result of Spènce (1960) for a = -1, w = 1 and b = 0; viz,

$$r = \sigma^{1/2}\left[1 + 0.0095\,(\sigma - 1)^2 + \cdots\cdots\right]$$

is also shown in the Fig. 3.4. It is seen that the Spence's result show a marked divergence from those of Pohlhausen for appreciably different from unity; thus, the region of validity of Spence's result is rather limited.

Our results (3.41), (3.57) and (3.70) for the heat transfer for the three cases - of low, high and order unity

Prandtl numbers - depends upon w, a and b, and also on mach number through the recovery enthalpy: the form of these results are quite complicated. However, we can get some insight by discussing the factors W_1, W_h and W_0 defined respectively by (3.42), (3,58) and (3.72). These quantities are proportional to heat transfer rate if dissipation is neglected. The contribution of dissipation may be added to respective W's to get the total heat transfer.

For low σ_∞ flows the quantity $W_1/\sigma_\infty^{1/2}$ vs. h_w is displayed in Fig. 3.6 for various values of m. It is seen that as m increases the quantity $W_1/\sigma_\infty^{1/2}$ decreases for $h_w < 1$, increases for $h_w > 1$ and remains zero for $h_w = 1$. This increase and decrease in $W_1/\sigma_\infty^{1/2}$ is more pronounced as we go away from $h_w=1$. The similar trend is observed in $W_0/\sigma_\infty^{1/3}$ vs. h_w for the Prandtl number of order unity as shown in Fig. 3.7. The last figure also displays $W_h/\sigma_\infty^{1/3}$ for high σ_∞ and it is seen that magnitude of $W_h/\sigma_\infty^{1/3}$ decreases as m increases. Lastly, our results (3.43) and (3.66) for the skin friction for low and order unity Prandtl numbers respectively differs by a numerical factor of 1.02, and the former is displayed in Fig. 3.8.

Thus we see that simple analysis of the type described here can be vary useful in showing with sufficient accuracy, the analytical dependence of the boundary layer characteristics on various parameters.

CHAPTER 4

HEAT TRANSFER IN FALKNER-SKAN BOUNDARY LAYER FLOWS

4.1 Introduction:

In previous Chapter 3, we obtained the leading terms in the asymptotic expansions, corresponding to the two limiting values of the Prandtl number of various quantities in the flow of a compressible fluid when pressure gradient is zero. In general, it is found that the zeroth order equations are coupled and nonlinear but those of higher order are coupled and linear. They all lead to two point boundary value problems with missing boundary conditions at either point (determined by matching). Therefore, the process of obtaining a few higher order terms in the perturbation expansion is quite labourious and involved. These higher order terms, as a matter of fact represent small corrections to the previously obtained terms. Therefore, it is desirable to develop at least some of them, in order to estimate theoretically the range of their validity. To accomplish the above, we consider here in this chapter, the similar two dimensional or axisymmetricfflow with pressure gradient at limiting values of the Prandtl number for the case of constant property fluids. In addition to limiting Prandtl number, the case when Prandtl number is of order unity is also studied.Heat transfer in Flakner-Skan flows has been studied by various

authors (as described in Chapter 1). Invariably all of them have either neglected the pressure gradient or dissipation.

4.2 Governing Equations:

The governing equations for the first order, steady plane or axisymmetric boundary layer flows of a constant property fluid as described in Sec. 2.5.1 are

$$(r^j u_1)_s + (r^j v_1)_N = 0 \tag{4.1a}$$

$$L_1(u_1) - u_{1NN} = U_1 U_{1s} \tag{4.1b}$$

$$L_1(h_1) - \sigma^{-1} h_{1NN} = E\left[u_1 U_1 U_{1s} - u_{1N}^2\right]. \tag{4.1c}$$

The last equation (4.1c) may be written in a more convenient fashion in terms of total enthalpy $g = (h_1 + E\,u_1^2/2)/g_e$. Adding Eu_1 times (4.1b) to (4.1c) we get,

$$L_1(g) - \sigma^{-1} g_{NN} = E(1 - \sigma^{-1})\,(u_1 u_{1N})_N \tag{4.1d}$$

and the corresponding boundary conditions are

$$\begin{aligned} N &= 0, \quad u_1 = 0 = v_1, \quad g = g_w \text{ or } g'(0) = 0 \\ N &= \infty, \quad u_1 \to 1, \quad g \to 1. \end{aligned} \tag{4.2}$$

Introducing the stream function ψ_1 defined by

$$\psi_{1N} = r^j u_1, \qquad \psi_{1s} = r^j v_1$$

the equation of continuity (4.1a) is automatically satisfied and if we assume the following form of ψ_1 and g

$$\Psi_1(s;\,N) = \sqrt{(2\xi)}\; f(\eta), \quad g(s,\,N) = g(\eta)\,. \tag{4.3}$$

with

$$\xi = \int_0^s U_1\, r^{2j}\, ds, \qquad \eta = r^j N/(2\xi)^{1/2}$$

the equation (4.1b) and (4.1d) reduce to the following total differential equations

$$f''' + ff'' + \beta\,(1 - f'^2) = 0 \tag{4.4}$$

$$g'' + \sigma f g' + C(\sigma - 1)\,(f' f'')' = 0\,. \tag{4.5}$$

Also the boundary conditions (4.2) become

$$f(0) = 0 = f'(0), \qquad f'(\infty) = 1 \tag{4.6a}$$

$$g(0) = g_w \text{ or } g'(0) = 0, \quad g(\infty) = 1. \tag{4.6b}$$

Here,

$\beta = (2\xi/U_1)\,(dU_1/d\xi)$ ~~is the pressure gradient~~ para-
is the pressure gradient parameter and the coefficient

$$C = \frac{(\gamma - 1)\, M_e^2}{2 + (\gamma - 1) M_e^2}$$

should be constants for the similarity to exist (M_e is the Mach number at the edge of boundary layer). The quantity C is indeed constant in the following three flows: (1) no pressure gradient, i.e., M_e = constant, (ii) incompressible $M_e = 0$ (iii) high Mach number $1/M_e \to 0$. However, in general C is a function of s. In our analysis we have retained C and treated

it as a constant. In the absence of more general results - i.e., without the use of similarity - one may calculate C using local value of M_e and hope that this will not lead to significant discripancy - particularly for the flows in which the properties do not vary rapidly.

The momentum equation (4.4) is the Falkner-Skan Boundary layer equation, and is independent of g and hence can be solved once and for all.

4.3 <u>Solution for Falkner-Skan Equation</u>:

The momentum equation (4.4) along with boundary conditions (4.6a) is the well-known Falkner-Skan (1930) equation. The solutions to this equation have been obtained by Hartree (1937) and recently improved by Smith (1954) for various values of β in the range β_s $(= -0.198838) \leq \beta \leq 2$. However, the solutions can be obtained for all $\beta > 0$. The mathematical existence of solution of (4.4) subject to (4.6a) has also been proved by Weyl (1942) for $0 \leq \beta \leq 1/2$. The uniqueness of solutions assuming they exist, has been established by Coppel (1960) for $\beta > 0$.

Further, for $\beta < 0$ the solutions are not unique and for every β in a certain range of values, it is possible that, there is a family of solutions (i.e., a continuous range of value of $f''(o)$) of (4.4) with $f(o) = 0 = f'(0)$ and $f' \to 1$ as $\eta \to \infty$. To exhibit this property we first introduce a new dependent variable

$$y(\eta) = 1-f'(\eta) \quad \text{or} \quad f = \eta - \alpha + \int_{\eta}^{\infty} y(x)\, dx \tag{4.7a}$$

in (4.4), we get,

$$y'' + (\eta - \alpha + \int^{\infty} y(x)\, dx\,)\, y' - \beta y\,(2-y) = 0. \tag{4.7b}$$

Now as we approach the edge of boundary layer, η becomes large and y becomes smaller, thereby,

$$\int_{\eta}^{\infty} y(x)\, dx \ll \eta \qquad \text{and } y(\eta) \ll 2$$

the equation (4.7b) degenerates to

$$y'' + (\eta - \alpha)\, y' - 2\beta\, y = 0 \tag{4.8}$$

or $$y \backsim \alpha_1 (\eta - \alpha)^{-2\beta-1} \exp((1/2)(\eta - \alpha)^2) + \alpha_2\, (\eta - \alpha)^{2\beta} \tag{4.9}$$

where α_1 and α_2 are undetermined constants.

For $\beta > 0$, it is clearly necessary to take $\alpha_2 = 0$, if the boundary condition $y\,(\eta \to \infty) = 0$ is to be satisfied, and then the solution is unique. However, for $\beta < 0$ the boundary condition $y(\eta \to \infty) = 0$ does not require α_2 to be zero. This nonvanishing of α_2 introduces a lack of uniqueness in the solution and in fact there exists a family of solutions of equation (4.4) satisfying (4.6a). To select an appropriate solution~~s~~ of equation (4.4) satisfying (4.6a) various arguments have been put forward. One is to suggest that the algebraic terms in (4.9), which introduces nonuniqueness arises essentially from the inertia terms and is therefore associated with the inviscid flow. So, it cannot arise with in the boundary layer and must be due to the boundary condition at the outer

edge. Another argument is based on the consideration of displacement thickness, i.e., if $f' \to 1$ algebraically and $\beta \geq -1$, the displacement thickness is infinite. Hence for $\beta < 0$, the condition at infinity to be used (as by Hartree 1937 and Smith 1954) is

$$\text{as } \eta \to \infty \qquad f' \to 1 \tag{4.10}$$

exponentially from below. In numerical work it is not easy to decide whether the asymptotic behaviour is purely exponential or contains a grain of algebra. However, it is reasonable to expect that with respect to the neighbouring solutions the exponential approach to be the most rapid.

It is shown by Hartree (1937) that there are no solutions of (4.4) subject to (4.6a) and (4.10) if $\beta < \beta_s$. Stewartson (1954) found the duality of solutions for $0 > \beta > \beta_s$, one of which has $f''(0) < 0$ and other $f''(0) > 0$; both exhibit the exponential behaviour (4.10) at infinity. Further, for the values of β slightly greater than β_s

$$f''(0) \propto (\beta - \beta_s)^{1/2}.$$

At $\beta = \beta_s$, the skin friction $f''(0)$ vanishes. Experiments at high, but finite, **values** of Reynolds number indicate that the seperation takes place at or very near the point of vanishing skin friction.

We now consider the solution to the Falkner-Skan equation as being known. We will be using them later for small and large η.

For small η, it is

$$f(\eta) = a\,\eta^2/2 - \beta\eta^3/3 + (2\beta-1)\,a^2\,\eta^5/5! + (4-6\beta)\,\beta a\,\eta^6/6! - 2(2-3\beta)\beta^2\,\eta^7/7! + (1-2\beta)\,(11-10\beta)\,a^3\,\eta^8/8! + O\,(\eta^9) \qquad (4.11)$$

and for large η is

$$f(\eta) = \eta - \alpha + f^*(\eta)\,,\quad f^*(\eta) \simeq 0\,(\eta^{-\infty}),\ \eta \to \infty. \quad (4.12)$$

Here $a = f''(0)$, α the displacement thickness and $\eta^{-\infty}$ means the exponentially small terms in the limit $\eta \to \infty$. The values of a and α for $\beta_s \leq \beta \leq 2$ are taken from Smith (1954).

4.4 Solution for high Prandtl Number:

Following section (3.3) we now study the energy equation (4.5) subject to (4.6b) at high σ, by the method of matched asymptotic expansions. Further, as shown later, it is necessary to study the problem seperately for the ordinary point ($f''(0) \neq 0$, $\beta \neq \beta_s$) and at separation point ($f''(0) = 0$, $\beta = \beta_s$). This is due to the fact that near $\beta = \beta_s$, the dominant term in the inner solution for $f(\eta)$ is of order of η^3 compared to η^2 at ordinary point. Thus the stretching required to form the inner variable for the energy equation (4.5) is of different order in the above two cases: $\beta = \beta_s$ and $\beta \neq \beta_s$. Lastly, it is shown that a logarithmic singularity is observed when $\beta = \beta_s$ in the solution.

4.4.1 Solution at Ordinary point ($\beta \neq \beta_s$):

a) Outer Limit: The outer limit is defined to be $\epsilon(=\sigma^{-1/3}) \to 0$, with η fixed; and assume the following outer expansion

$$g = \sum_{n=0} g_n \epsilon^n . \tag{4.13}$$

Substituting the expansion (4.13) in (4.5) and collecting the terms of various powers of ϵ, we get,

$$\begin{aligned} fg_0' &= -C(f'f'')' \\ fg_1' &= 0, \ fg_2' = 0, \\ fg_3' &= -g_0'' + C(f'f'')' \\ fg_r &= -g_{r-3}'' , \qquad r = 4,5 \ldots \end{aligned} \tag{4.14}$$

The solutions to first four equations in (4.14), which satisfy the boundary conditions at infinity are,

$$\begin{aligned} g_0 &= 1 + C\int_\eta^\infty (f'f'')'/f \, d\eta \\ &= 1 + C(f'^2-1)/2 + C\int_\eta^\infty J/f \, d\eta \\ g_1 &= 0, \qquad g_2 = 0, \\ g_3 &= -C\int_\eta^\infty (J/f)'/f \, d\eta \end{aligned} \tag{4.15}$$

where $J(\eta) = f''^2 + \beta f'(1 - f'^2)$.

Now this solution for g shows that the total enthalpy g increases as $\eta \to 0$. Thus this solution not only fails to satisfy the boundary condition at the wall, but is singular there. To study the detailed behaviour of this singularity

near the wall we first write the inner expansion of the outer solution. The behavior of the integrand J/f as $\eta \to 0$ is (using 4.11),

$$J/f \simeq 2a/\eta^2 - 16\beta/(3\eta) + O(\eta^3), \quad a \neq 0 \quad . \tag{4.16a}$$

Thus (4.15) indicates that the total enthalpy is singular like η^{-1} and $\ln \eta$ as $\eta \to 0$. Now to make the integral $\int_\eta^\infty J/f \, d\eta$ bounded we have to add and substract singular part (4.16a). Thus the integral

$$\int_\eta^\infty J/f - 2a^2/\eta + 16\beta/3\eta \quad d\eta$$

is regular near $\eta = 0$, but this integral diverges at upper limits. To avoid this difficulty, we can combine the second and third terms of (4.16) binomially as $\eta \to 0$, which is in some sense equivalent to writing

$$J/f \simeq 2a/\eta^2 - 16\beta/(3\eta(\eta+1)) \; . \tag{4.16b}$$

The inner expansion of the outer solution (4.15) as $\eta \to 0$ may be written as

$$g = 1-C/2 + C\, I_\beta(0) + 2Ca/\eta + (16\beta C/3)\ln\eta + \ldots$$
$$\ldots + O(\epsilon^3 \ln\eta), \tag{4.17}$$

where

$$I_\beta(\eta) = \int_\eta^\infty J/f - 2a/\eta^2 + 16\beta/(3\eta^2 + 3\eta)\; d\eta$$

is regular near $\eta = 0$.

b. Inner Solution: It is now clear from (4.17) that a different inner limit is needed for small η. From an order of magnitude

analysis we introduce the inner variables

$$y = \eta / \epsilon \quad , \qquad \breve{\theta} = \epsilon g \tag{4.18}$$

and study the limit as $\epsilon \to 0$ with and θ fixed. From (4.18) and the inner limit of the outer solution (4.17), it is obvious that the logarithms should appear in the inner expansion

$$\breve{\theta} = \breve{\theta}_0 + \sum_{n=1} (\epsilon \ln \epsilon \, \breve{\theta}_{2n-1} + \epsilon^n \breve{\theta}_{2n}) . \tag{4.19}$$

Substituting (4.18), (4.19) and (4.11) in (4.5) and collecting the coefficients of like powers of ϵ; we get

$$\begin{aligned}
&Q(\breve{\theta}_0) + Ca^2 = 0 \\
&Q(\breve{\theta}_1) = 0 \\
&Q(\breve{\theta}_2) - \frac{\beta}{3!} y^3 \breve{\theta}_{oy} - 3 \beta a C y = 0 \\
&Q(\breve{\theta}_3) - \frac{\beta}{3!} y^3 \breve{\theta}_{1y} = 0 \\
&Q(\breve{\theta}_4) - \frac{\beta}{3!} y^3 \breve{\theta}_{2y} + \frac{3}{2} C\beta^2 y^2 = 0 \\
&Q(\breve{\theta}_5) - \frac{\beta}{3!} y^3 \breve{\theta}_{3y} = 0 \\
&Q(\breve{\theta}_6) - \frac{\beta}{3!} y^3 \breve{\theta}_{4y} + (2\beta-1) \frac{a^2}{5!} y^5 \breve{\theta}_{oy} \\
&\qquad\qquad + \frac{5C}{6} (2\beta-1) a^3 y^3 - Ca^2 = 0 \\
&Q(\breve{\theta}_7) - \frac{\beta}{3!} y^3 \breve{\theta}_{5y} = 0
\end{aligned} \tag{4.20}$$

where Q is a differential operator defined as

$$Q(\,) = d^2/d y^2 + \frac{a}{2!} y^2 \; d/dy \; .$$

Each of the above equations is linear differential equation of second order. In present case each of the above can be solved in terms of gamma and related gamma functions. However, as the solutions of the higher order equations require the solutions of the subsequent lower order equations, it is very laborious to solve many higher order equations. The solution to the first seven equations is given below

$$
\begin{aligned}
\breve{\theta}_0 &= M_0 - 12\, Ca^2\, N_{-1/3}\,(1/3,\ x)/(36a)^{2/3} \\
\breve{\theta}_1 &= M_1 \\
\breve{\theta}_2 &= M_2 + \beta b_0 \quad (5/3,x)/(48a^5)^{1/3} - \beta C(6N_0(2/3,x) \\
&\qquad - N_1\,(1/3,\ x) + N_1(5/3,x)/5)/2 \\
\breve{\theta}_3 &= M_3 + \beta b_1 \quad (5/3,x)/(48a^5)^{1/3} \\
\breve{\theta}_4 &= M_4 - \beta b_2 \quad (5/3,x)/(48a^5)^{1/3} - \beta^2 b_0 \quad (3,x)/(16a^3) \\
&\qquad + \beta^2 C\,(6/a^4)^{1/3}\ N_{7/3}(1/3,x)/16 \\
&\qquad + N_{7/3}\,(3,x)/144 - N_{7/3}(5/3,x)/40 \\
&\qquad - 3N_{4/3}(2/3,x)/4 - N_{4/3}(2,x)/4 - N_{1/3}(1,x)) \qquad (4.21) \\
\breve{\theta}_5 &= M_5 + \beta b_3 \quad (5/3,x)/(48a^5)^{1/3} + \beta^2 b_1 \quad (3,x)/(16a^2) \\
\breve{\theta}_6 &= M_6 + \beta b_3 \quad (5/3\ x)/(48a^5)^{1/3} + \beta^2 b_2 \quad (3,x)/(16a^2) \\
&\qquad + \beta^3 b_0 \quad (13/3,x)/(32(36a^{13})^{1/3}) \\
&\qquad + (1-2\beta)b_0 \quad (7/3,x)/(10(36a)^{1/3}) \\
&\qquad + 2C\,(a^4/6)^{1/3}\ N_{-1/3}\,(1/3,x) \\
&\qquad + C(1-2\beta)(a^4/6)^{1/3}\ 5N_{\frac{2}{3}}(4/3,x)/2
\end{aligned}
$$

$$-N_{5/3}(1/3,x)/10 + N_{5/3}(7/3,x)/70)-C\beta^3(12N_{5/3}(1,x)$$
$$-36N_{5/3}(7/3,x)/7 - 9N_{8/3}(2/3,x)/2 + 3N_{8/3}(2,x)$$
$$-9N_{8/3}(10/3,x)/10 + N_{11/3}(1/3,x)/4- 3N_{11/3}(5/3,x)/20$$
$$+N_{11/3}(3,x)/12 - N_{11/3}(13/3,x)/52)/(3072a^8)^{1/3}.$$

In writing down the above solutions we have used the relation

$$\int_0^y z^n \exp(-az^r/r!) \int_0^z t^n \exp(at^r/r!)dt\, dz$$
$$= (r!/a)^{(m+n+2)/r} N_{(m+n+2-r)/r}\left[(m+n+2)/r, ay^r/r!\right]/(r+rm)$$

where $N_m(\alpha,x) = \int_0^x t^m N(\alpha,t)dt$, $N(\alpha,x) = \alpha\int_0^1 e^{-xt}(1-t)^{\alpha-1}\, dt$ (4.22)

are certain related gamma functions (see Appendix A). Further, $\gamma(\alpha,x)=\int_0^x e^{-t}\, t^{\alpha-1}\, dt$ is the well-known incomplete gamma function.

and $x = (a/3!)y^3$, $M_r = a_r + 2b_r\, \gamma(1/3,x)/(36a)^{1/3}$, $r=0,1\ldots$, (4.23)

and a_r and b_r are constants of integration to be determined from the inner boundary conditions and by matching.

c Results: Various unknown constants in (4.21) and (4.18) will now be determined by the use of matching principle.

A. Insulated Wall Case: Here the inner boundary condition $g'(0) = 0$ requires $b_0 = b_1 = \ldots = 0$ and the matching principle gives the recovery factor

$$r_f = (2g_r-2+C)/2$$
$$= 4(a^4/6)^{1/3}\, n(-1/3,1/3)\,\sigma^{1/3} - 32\beta \ln\sigma/9$$
$$+ 2I_\beta(0) - \beta\, m_1 - 32\beta \ln (a/6)/9$$
$$+ 2\beta^2(6/a^4)^{1/3}\sigma^{-1/3}\left[(n(7/3,1/3)/16+n(7/3,3)/144\right.$$
$$-n(7/3,5/3)/40-3n(4/3,2/3)/4 +n(4/3,2)/4$$

$$+ n(1/3,1) \,)\Big] + \sigma^{-2/3}\Big[-4(a^4/6)^{1/3} n(-1/3,1/3)$$
$$+ (1-2\beta)(a^4/6)^{1/3} \quad (-5n(2/3,4/3) + n(5/3,1/3)/5$$
$$-n(5/3,7/3)/35) - \beta^3(12n(5/3,1)-36n(5/3,7/3)/7$$
$$-9n(8/3,2/3)/2 + 3n(8/3,2)-9n(8/3,10/3)/10$$
$$+n(11/3,1/3)/4 - 3N(11/3,5/3)/20 + n(11/3,3)/12$$
$$-n(11/3,7/3)/52/(384a^8)^{1/3}\Big] + O(\sigma^{-1} \ln\sigma).$$

Where $m_1 = \int_0^\infty 6N(2/3,x)-xN(1/3,x)+xN(5/3,x)/5-32/(x^2+x) \quad dx$

and $n(m,\alpha) = N_m(\alpha, \infty)$. This can be evaluated in terms of gamma function from (4.22) by changing the order of integration.

For the case of no pressure gradient $\beta = 0$, the above result (4.24) becomes,

$$r_f = 1.922\,\sigma^{-1/3} - 1.341 + 0.468\,\sigma^{-2/3} + \ldots \qquad (4.25)$$

Here in the above expression, the first term is due to Meksyn (1961) and first two due to Narasimha and Vasantha (1966).

B. Heat Transfer Case: In this case the inner boundary condition $g(0) = g_w$ requires $a_o = a_1 = 0$, $a_2 = g_w$, $a_3 = a_4 = \ldots = 0$. The matching principle gives the heat transfer rate at the wall as

$$g'(0) = (g_r - g_w)\, D\,(\beta,\sigma)$$

where

$$D(\beta,\sigma) = 0.61627\,(a\sigma)^{1/3} - 0.09434\,\beta/a$$
$$-0.03303(\beta^2/a^7)\,\sigma^{-1/3} + \Big[0.013695\,(2\beta - 1)$$
$$-0.020966\,\beta^3/a^4\Big]\, a^{1/3}\sigma^{-2/3} + O(\sigma^{-1})\,. \qquad (4.26)$$

4.4.2 Solution at Point of Vanishing Skin Friction:

At the point of vanishing skin friction, the asymptotic relations just derived in (4.4.1) do not apply, as explained in the beginning of this section. The correct expansions can again be derived by the present method.

a) Outer limit: Following, section (4.4.1) we define the outer limit as $\sigma \to \infty$ with η fixed, writing

$$g = \Sigma g_n \sigma^{-n} \tag{4.27}$$

in (4.5), we get

$$\begin{aligned} fg_o' &= -C(f'f'')' \\ fg_1' &= -g_o'' + C(f'f'')' \\ fg_r' &= -g_{r-1}'' , \quad r = 2,3, \ldots . \end{aligned} \tag{4.28}$$

The solutions to the first two equations which satisfy the boundary condition at infinity are

$$\begin{aligned} g_o &= 1 + \frac{C}{2}(f'^2 - 1) + C \int_\eta^\infty J/f \, d\eta \\ g_1 &= -C \int_\eta^\infty (J/f)'/f \, d\eta . \end{aligned} \tag{4.29}$$

These solutions also fails at satisfy the boundary condition at the wall. To write the inner expansion of this outer solution (4.35), we first evaluate the integrands from (4.11) as $\eta \to 0$

$$J/f \approx -9\beta/\eta - \frac{2244}{4!}(2+3\beta)\beta^2\eta^3/4! + O(\eta^7)$$

$$(J/f)'/f \approx -\frac{54}{\eta^5} + (51 + 171\beta)\frac{\beta}{7\eta} + O(\eta^3).$$

Employing, the same arguments as in (4.4.1a), the inner expansion of outer solution (4.35) is seen to be

$$g(\eta \to 0) = + \alpha 9\beta C \ln\eta + 1 - C + C\bar{I}_1(0) + $$
$$+ \sigma^{-1}\left[+(51-171\beta)\beta C \ln\eta /7 - C\bar{I}_2(0) + \frac{27C}{2\eta^4}\right]$$
$$+ O(\sigma^{-2}\ln\eta) \qquad (4.30)$$

where

$$\bar{I}_1(\eta) = \int_{\eta}^{\infty} J/f + 9\beta/\eta \; d\eta$$
$$\bar{I}_2(\eta) = \int_{\eta}^{\infty} (J/f)'/f + 54/\eta^5 - \frac{\beta(51-171\beta)}{7\eta(\eta+1)} \; d\eta$$

are regular near $\eta = 0$.

b. Inner Solution: To get the inner solution, we first need the proper inner variables. From an order of magnitude analysis the inner independent variable is found to be $\bar{z} = \eta\sigma^{1/4}$. However, if z is introduced in (4.36), then the leading term starts with $\ln\eta$ and this gives the proper inner variables as

$$\bar{z} = \eta\sigma^{1/4}, \qquad \breve{\theta} = g/\ln\sigma . \qquad (4.31)$$

we now define the inner limit as $\sigma \to \infty$ with θ and z fixed, and writing

$$\breve{\theta} = \sum_{n=0} (\breve{\theta}_{2n} + \breve{\theta}_{2n+1}/\ln\sigma)\sigma^{-n} \qquad (4.32)$$

we get

$$Q_1(\breve{\theta}_o) = 0$$

$$Q_1(\breve{\theta}_1) + \frac{3C}{2}\beta^2 \bar{z}^2 = 0$$

$$Q_1(\breve{\theta}_2) - \frac{2}{7!}(2-3\beta)\,\beta^2\,\bar{z}^7\,\theta_{o\bar{z}} = 0$$

$$Q_1(\breve{\theta}_3) - \frac{2}{7!}(2-3\beta)\,\beta^2\bar{z}^7\,\theta_{1\bar{z}} + 56C(2+3\beta)\,\beta^3\,\bar{z}^6/6!$$
$$- 3C\beta^2\bar{z}^2/2 = 0 .$$

Here Q_1 is a differential operator, defined as

$$Q_1 = d^2/d\bar{z}^2 - \frac{\beta\bar{z}^3}{3}\,d/d\bar{z} .$$

The solutions to above equations are

$$\breve{\theta}_o = M_{oo}$$
$$\breve{\theta}_1 = M_{11} + 3C\beta\,N_o\,(3/4,\,x)$$
$$\breve{\theta}_2 = M_{22} + b_o(2-3\beta)\,(24/-\beta)^{1/4}\,\gamma(\tfrac{9}{4},\,x)/140$$
$$\breve{\theta}_3 = M_{33} + C\beta_1(2-3\beta)\left[8N_1\,(7/4,\,x) - 3/7\,N_2(3/4,\,x) + 9/77\,N_2\,(11/4,\,x)\right]/\,5 - 3C\beta\,N_o(3/4,x)$$
$$+ b_1(2-3\beta)\,(24/-\beta)^{1/4}\gamma(9/4,x)/140$$

where

$$x = -\beta\bar{z}^4/4!$$

$$M_{rr} = a_r + \left\{b_r(24/\beta)^{1/4}\right\}\gamma(1/4,\,x)\,/\,4$$

a_r and b_r are constants of integration.

c. Results:

a) Insulated wall case: In this case $b_o = b_1 = \ldots = 0$ and the matching the inner and outer solutions we get the enthalpy recovery factor is

$$r_f = -9\beta \ln\sigma/2 - \beta\, m_2 - 9\beta h(-\beta/24) + 2I_1(0) \qquad (4.35)$$

$$- (51-171\beta)\, \beta\sigma^{-1} \ln\sigma /14$$

$$+ \Big[6\, m_2 + C\beta\, (2-3\beta)m_3/5 + \beta(51-171\beta)\, \ln(-\beta/24)/48$$

$$- 2J_2\, (0)\Big]\sigma^{-1} + O(\sigma^{-2} \ln\sigma)$$

where $m_2 = \int_0^\infty N(3/4,\ x) - 3/(4x + 4)\ dx$

$$m_3 = \int_0^\infty 8xN(7/4,\ x) - 3x^2\, N(3/4,\ x)/7 + 9x^2 N(11/4,x)/77$$

$$- 187/14 + \frac{285}{28(x+1)}\ dx\ .$$

b) Heat transfer case: Here the inner boundary condition $g(0) = g_w$ requires $a_o = a$, $a_1 = g_w$, $a_2 = a_3 = 0$, and the matching principle gives the heat transfer rate at the wall

$$g'(0) = (g_r - g_w)\, D\, (\beta_s, \sigma)$$

where

$$D(\beta_s, \sigma) = \frac{\sigma^{-1/4}}{B_{oo}} \left[1 - \frac{B_{1o}}{B_{oo}}\ \sigma^{-1} + O(\sigma^{-2})\right]$$

$$B_{oo} = \frac{1}{4} \left(\frac{24}{-\beta}\right)^{1/4} \Gamma\!\left(\frac{1}{4}\right)$$

$$B_{1o} = (2 + 3\beta)\left(\frac{24}{-\beta}\right)^{1/4} \Gamma\!\left(\frac{9}{4}\right) / 140\ .$$

At $\beta = \beta_s = -0.198838$ the above result reduces to

$$D(\beta_s, \sigma) = 0.33285\ \sigma^{1/4} - 0.007716\ \sigma^{-3/4} + O(\sigma^{-7/4}). \qquad \ldots (4.36)$$

4.5 Solution at Low Prandtl Number:

In this section, we study the solution of the energy equation (4.5) along with (4.6b) for small values of σ. The procedure followed is similar to that of (4.4.1).

4.5.1 Inner limit: First we define an inner limit as the process $\sigma \to 0$ with η fixed, and write

$$g = \sum_{n=0} g_n \sigma^{n/2} . \tag{4.37}$$

Substituting (4.51) in (4.5) and collecting the coefficients of the like powers of σ, we get

$$\begin{aligned}
g_o'' &= C(f'f'')' \\
g_1'' &= 0 \\
g_2'' &= fg_o' - C(f'f'')' \\
g_r'' &= -f\,g_{r-2}' , \qquad r = 3,4,\ldots
\end{aligned} \tag{4.38}$$

The solutions to first six equations (4.38) are

$$\begin{aligned}
g_o &= Cf'^2/2 + C_o\eta + d_o, \\
g_1 &= C_1\eta + d_1, \\
g_2 &= -C_o\left[\eta^3/6 - \eta^2\,\alpha/2 + \int_o^\eta (\eta - x)\, f^*(x)\, dx\right] \\
&\quad - C\int_o^\eta (\eta - x)\, J(x)\, dx + C_2\eta + d_2, \\
g_3 &= -C_1\left[\eta^3/6 - \eta^2\,\alpha/2 + \int_o^\eta (\eta - x)\, f^*(x)\, dx\right] + \\
&\quad + C_3\eta + d_3
\end{aligned}$$

$$
\begin{aligned}
g_4 = {} & C(\eta^3/6 - \eta^2\alpha/2) \int_0^\eta J(x)\,dx - C\int_0^\eta \left(\frac{x^2}{2} - x\alpha\right)J(x)dx \\
& - C\int_0^\eta (x^3/3 - x^2\alpha/2)\,J(x)\,dx + C\int_0^\eta (\eta - x)\,f^*(x)\int_0^x \\
& J(t)\,dt\,dx + C_0\Big[\eta^5/40 - \eta^4\alpha/8 + \eta^3\alpha^2/6 \\
& + (\eta^3/6 - \eta^2\alpha/2)\int_0^\eta f^*(x)\,dx \\
& + \int_0^\eta (\eta - x)\,f^*(x)\int_0^x f^*(t)\,dt\,dx \\
& - \int_0^\eta (x^3/6 - x^2\alpha/2)\,f^*(x)\,dx\Big] - C_2\Big[\eta^3/6 - \eta^2\alpha/2 \\
& + \int_0^\eta (\eta - x)\,f^*(x)\,dx\Big] + C_4\eta + d_4 \\
g_5 = {} & C_1\Big[\eta^5/40 - \alpha\,\eta^4/8 + \alpha^2\eta^3/6 \\
& + (\eta^3/6 - \alpha\,\eta^2/2)\int_0^\eta f^*(x)\,dx \\
& + \int_0^\eta (\eta - x)\,f^*(x)\int_0^x f^*(t)\,dt\,dx - \\
& - \int_0^\eta (x^3/6 - x^2\alpha/2)\,f^*(x)\,dx\Big] - C_3\Big[\eta^3/6 - \alpha\eta^2/2 \\
& + \int_0^\eta (\eta - x)\,f^*(x)\,dx\Big] + C_5\eta + d_5
\end{aligned}
$$

where f^* is defined as

$$f^*(\eta) = \alpha - \eta + f(\eta), \quad f^* \simeq O(\eta^{-\infty}) \text{ as } \eta \to \infty,$$

and $$J(\eta) = f''^2 + \beta f'(1 - f'^2)\,. \tag{4.40}$$

It is seen from the above that the solutions become singular at large η and do not satisfy the boundary condition at infinity. This singularity is similar to one encountered in Stokes solution in low Reynolds number flows.

4.5.2 Outer limit: It is clear from (4.55) that a different outer limit is needed for large η. From an order of magnitude analysis we introduce the outer variable

$$\zeta = \sigma^{-1/2}\eta \tag{4.41}$$

and study the limit $\sigma \to 0$ with ζ fixed.

From (4.12) for large , we have

$$f(\eta) = \eta - \alpha + O(\eta^{-\infty})$$

and the equation (4.5) is present outer limit reduces to the following outer flow equation

$$g_{\zeta\zeta} + (\zeta - \sigma^{-1/2}\alpha)\ g_{\zeta} = O(\sigma^{\infty}) . \tag{4.42}$$

This equation is correct to all order in σ, i.e., the error here is exponentially small, as explained in section (3.1.1b). The outer equations of all orders may be obtained in a routine way from this simple equation. But this equation can be solved once and for all. The solution to equation (4.52) which satisfies $g(\infty) = 1$ is

$$g = A + (1 - A)\ \text{erf}\left[(\zeta - \alpha\sigma^{-1/3})/\sqrt{2}\right] \tag{4.43}$$

where erf (x) is the error function defined as

$$\text{erf}\ (x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\ dt$$

and A is a constant of integration, to be expanded in powers of $\sigma^{-1/2}$ instead of solution,

$$A(\sigma) = \sum_{n=0} A_n \sigma^{-n/2}.$$

Either the constants C's or d's can be determined from the inner boundary conditions and the others along with D's are to be determined by the application of the matching principle.

4.5.3 Results:

a. Insulated wall case: For a thermally insulated surface $g'(0) = 0$, so $C_0 = C_1 = \dots C_5 = 0$, and the matching principle gives the enthalpy recovery factor

$$\begin{aligned} r_f = {} & \sqrt{(2\pi\sigma)} \int_0^\infty J(x)dx + 2\sigma \int_0^\infty (\alpha-x)\, J(x)dx \\ & + \sqrt{(\pi\sigma^3/2)} \Big[(\alpha^2 - 2\int_0^\infty f^*(x)\, dx) \int_0^\infty J(t)\, dt \\ & + 2\int_0^\infty J(x) \int_0^x f(t)\, dt\, dx \Big] + 2\sigma^2 \Big[(\frac{\alpha^3}{3}) \int_0^\infty J(x)\, dx \\ & + \int_0^\infty J(x) \int_0^x (\alpha-t)\, f(t)\, dt\, dx \\ & + \int_0^\infty (\alpha-x)\, f^*(x)\, dx \int_0^\infty J(t)\, dt \Big] + O(\sigma^{5/2}). \end{aligned} \tag{4.44}$$

b. Heat transfer Case: In this case we prescribe a constant wall enthalpy $g(0) = g_w$ to study the heat transfer at the surface. In order to satisfy this inner boundary condition we must have $d_0 = g_w$, $d_1 = d_2 = \dots = 0$. Further, the matching gives the heat transfer rate at the wall

$$g'(0) = (g_r - g_w)\, D(\beta,\sigma)$$

where

$$\begin{aligned} D(\beta,\sigma) = {} & (2\sigma/\pi)^{1/2} - \frac{2\alpha}{\pi}\sigma + (2\sigma^3/\pi)^{1/2} \Big(\int_0^\infty f^*(t)\, dt \\ & + \frac{4-\pi}{2\pi}\alpha^2 \Big) + \sigma^2 \Big(-\frac{2}{\pi} \int_0^\infty (\alpha+x) f^*(x)\, dx + \frac{4(\pi-3)}{3\pi^2}\alpha^3 \Big) \\ & + O(\sigma^{5/2}). \end{aligned} \tag{4.45}$$

Here if dissipation is neglected i.e. $g_r = 1$, the first three terms in (4.45) are due to Goddard and Acrivos (1966).

4.6 Solution for Prandtl Number of Order Unity:

To solve the energy equation (4.5) along with the boundary conditions (4.6b), when Prandtl number is of order unity, we use the method of inversion of variables developed by Meksyn (1961). The equations (4.4) and (4.5) are integrated respectively once and twice to give

$$f'' = y(\eta, \beta) \exp(-\bar{F}) \tag{4.46}$$

$$g = \bar{A} + \bar{B} \int_0^\eta \exp(-\sigma\bar{F})\, dy - C(\sigma - 1) \int_0^\eta \exp(-\sigma\bar{F})\, dx \int_0^x y_1(t) \exp((\sigma-2)\bar{F})\, dt \tag{4.47}$$

where $\bar{F} = \int_0^\eta f(x)\, dx$

$$y = a - \beta \int_0^\eta (1 - f'^2(t))\, dt \tag{4.47a}$$

$$y_1 = (1 + f' f'''/f''^2)\, y^2 \,.$$

Here the constants of integration $\bar{A}$ and $\bar{B}$ are to be evaluated from the boundary conditions.

For an insulated wall, the condition $g'(0) = 0$, requires $\bar{B} = 0$ and using the boundary condition at infinity $g(\infty) = 1$, the equation (4.47) becomes

$$g = 1 + C(\sigma-1) \int^\infty \exp(-\sigma\bar{F})\, dt \int_0^t y_1 \exp((\sigma-2)\bar{F})\, dx. \tag{4.48}$$

The recovery factor is

$$r_f = 1 + 2(\sigma-1) \int_0^\infty \exp(-\sigma\bar{F})\, dt \int_0^t y_1 \exp((\sigma-2)\bar{F})\, dx \,. \tag{4.49}$$

Next, when the wall enthalpy is prescribed, i.e. $g(0) = g_w = \bar{A}$. In view of the boundary condition at infinity, the solution (4.47) becomes

$$g(\eta) = g_w + (g_r - g_w) \int_0^\eta \exp(-\sigma\bar{F})\,d\eta \Big/ \int_0^\infty \exp(-\sigma\bar{F})\,d\eta$$
$$-C(\sigma-1)\int_0^\eta \exp(-\sigma\bar{F})\,dt \int_0^t y_1 \exp((\sigma-2)\bar{F})\,dx\,, \tag{4.50}$$

Heat transfer rate at the wall is

$$g'(0) = (g_r - g_w)/HT, \quad HT = \int_0^\infty \exp(-\sigma\bar{F})\,d\eta\,. \tag{4.51}$$

Now to evaluate the integrals (4.48) and (4.51) we need η in terms of $\bar{F}$. To obtain this relationship, we follow the usual procedure by assuming a power series of the type (4.11) for f in terms of η. Now with this series (4.11), the expressions (4.47a) become

$$\bar{F} = (a\eta^3/3!) - \beta\eta^4/4! - a^2(1-2\beta)\eta^6/6! + 2a\beta(2-3\beta)\eta^7/7!$$
$$-\beta^2(4-6\beta)\eta^8/8! + O(\eta^9) \tag{4.52}$$

$$y = a - \beta\eta + 2\beta a^2\eta^3/3! - \beta a(1+6\beta)\eta^4/4! \tag{4.52}$$
$$+ \beta^2(1+6\beta\,\eta^5/5! + 10\beta(a^3(1+2\beta\,\eta^6)/6! + O(\eta^7)$$

$$y_1\exp(-2\bar{F}) = a^2 - 3a\beta\eta + 3\beta^2\eta^2/2 - 5a^3(1-2\beta)\eta^3/6 + \cdots\,.$$

Inverting the first of the (4.52) we obtain the following power series for η in terms of $\bar{F}$

$$\eta = \sum_{m=0} \frac{A_m}{m+1}\bar{F}^{(m+1)/r}\,. \tag{4.53}$$

In (4.53) $r = 3$ or 4, accordingly as $a \neq 0$ or $a = 0$. So, these two cases are considered rseparately.

4.6.1 Ordinary point Case $a \neq 0$: In this case $r = 3$ and the relation (4.53) becomes

$$\eta = (6\bar{F}/a)^{1/3} + \beta(6\bar{F}/a)^{2/3}/(12a) + \beta^2\bar{F}\,(8a^3)$$
$$+(6/a)^{1/3}\left[\frac{(1-2\beta)}{60} + \frac{35\beta^3}{864a^4}\right]\bar{F}^{\,4/3} + \cdots .$$

Expressing y_1 in terms of $\bar{F}$ we get

$$y_1 = a^2 - 3\beta\,(6\bar{F}a^2)^{1/3} + 5\beta^2(6\bar{F}/a)^{2/3}/4 + \cdots . \qquad (4.54)$$

Substituting (4.54) in (4.49) and using the relation (3.47) we get the following expression for the recovery factor.

$$r_f = 1 + 2(\sigma-1)\Big[3(36a^4)^{1/3}\,\Gamma(2/3)\;{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{\sigma-2}{\sigma})\,\sigma^{-2/3}$$
$$-9\beta\left\{4\,{}_2F_1(1/3,\ 1,\ 4/3,\ \tfrac{\sigma-2}{\sigma}) - {}_2F_1(\tfrac{2}{3}, 1, \tfrac{5}{3}, \tfrac{\sigma-2}{\sigma})\right\}\sigma^{-1}$$
$$-(\beta/8)(6/a^4)^{1/3}\,\Gamma(\tfrac{1}{3})\left\{39\,{}_2F_1(1/3, 4/3, 4/3, \tfrac{\sigma-2}{\sigma})\right.$$
$$\left.+\ 8\,{}_2F_1(2/3, 4/3, 5/3, \tfrac{\sigma-2}{\sigma})/3 + {}_2F_1(1, \tfrac{4}{3}, 2, \tfrac{\sigma-2}{\sigma})\right\}\sigma^{-4/3} + \cdots\Big].$$

For the case of no pressure gradient $\beta = 0$, the above result is

$$r = 1+6(\sigma-1)(36a^4)^{1/3}\,\Gamma(\tfrac{2}{3})\;{}_2F_1\,(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{\sigma-2}{\sigma})\sigma^{-2/3} + \cdots . \qquad (4.55)$$

Similarly the heat xtransfer result (4.51) reduces to

$$HT = (g_r - g_w)/g'(0)$$
$$= 1.62265\,(a/\sigma)^{1/3} - 0.248400\,(\beta/a^{5/3})$$
$$+0.12500\,\beta^2\,\sigma^{-1/3}/a^3 - \Big[0.08764\,\beta^3/a^4 +$$
$$+0.03606\,(1-2\beta)\Big](a^{-1}\sigma^{-2})^{1/3} + O\,(\sigma^{-1}).$$

4.6.2 Case of separation point $a = 0$: For the present case when $a = 0$, $\beta = \beta_s$, $r = 4$ and the equation (4.53) becomes

$$\Upsilon = (24\bar{F}/-\beta)^{1/4}\Big[1+(2-2\beta)\ \bar{F}/1+0 + (3\beta+2)(202+ + 145\beta)\ \bar{F}^2/77160 + \ldots\Big]. \tag{4.57}$$

Further the expression for y_1 becomes

$$y_1 = (-24\beta^3\bar{F})^{1/2}\Big[3/2 + (465\ \beta - 64)\ \bar{F}/82 + \ldots\Big]. \tag{4.58}$$

Now using (4.57) and (4.58) the result (4.49) for the recovery factor becomes

$$r = 1 + 57.2653 \quad {}_2F_1(1/4,\ 1,\ 5/4,\ \frac{\sigma-2}{\sigma})(1-\sigma^{-1}) + \ldots \ldots \tag{4.59}$$

and the heat transfer result (4.51) become

$$HT = 3.0043\ \sigma^{-1/4} + 0.06965\ \sigma^{-5/3} -0.004899\ \sigma^{-9/4} + 0(\sigma^{-13/4}).$$

4.7 <u>Discussion</u>

The main results of interest in this Chapter are (4.26, 4.36), (4.45) and (4.56, 4.60) for the heat transfer and (4.24, 4.35), (4.44) and (4.55, 4.59) for the recovery factor respectively for the cases of high, low and order unity values of Prandtl number.

The heat transfer results (4.26, 4.36) for high σ are shown in Fig. 4.1. The convergence of high σ series (4.26) is good even up to $\sigma = 0.5$ with $-1 < \beta \leq 2$. In the region $\beta \leq -1$ the convergence of the series (4.26) becomes poor owing to the

was not always successful, an extension described by Hardy (1949) and called by him the (E, q) process gave very satisfactory results. In this method whose application is described in Appendix A-2, we must choose a suitable q, possibly by trial and error procedure (in conventional Eulerisation process as q is taken as unity) which gives the desired convergence.

We can improve the convergence of either the low or the high σ or both the series in the intermediate range. In the present work we have Eulerised the low σ series (as its convergence is poor compared to high σ results) and the results are shown in Fig. 3.4. The results are extremely good in the sense that the gap is filled even for $\beta = -0.16$ and -0.198838.

In the above para the low σ series is Eulerised in the region $0.1 < \sigma < 1$. However, the series can be Eulerised even for much higher values of σ. To illustrate this we again study the no pressure gradient case and compare the present results with those given by various authors in Fig. 4.5. If the first term in low σ series is used as given by Stewartson (1964) there is an overestimate of D(0, 0.25) by 40% . If two terms are used as given by Morgan, Pipkin and Warner (1958), there is an overestimate of D(0, 0.25) by 28%. No comparison is made with the three term result of Goddard and Acrivos (1966) as there is a numerical error in their third term for $\beta = 0$. Further, the result of Sparrow (1955).

$$D(0,\sigma) = 0.748\,\sigma^{1/2}/(1 + 0.82\,\sigma^{1/2}), \quad \sigma \to 0,$$

based upon the Karman-Pohlhausen method underestimate $D(\beta,\sigma)$, throughout the range, by 6 % at small values of σ and by 12% at $\sigma = 1$. It is interesting to note that Eulerised low σ results pass through the exact results of Pohlhausen (1921) even for $\sigma = 15$, and for σ of this order overlap with only the one term in high σ result (4.26) due to Narasimha and Vasantha (1966). However, to eulerise low σ series beyond this point, we need more number of terms to get useful results even with large values of q.

Regarding the heat transfer results (4.56, 4.60) for Prandtl number of order unity we wish to say as follows. If we just expand $D(\beta,\sigma)$ series (4.56, 4.60) binomially for large we arrive at results similar to (4.26, 4.36) obtained for large σ. It does not mean that the heat transfer = $D(\beta,\sigma)\,(g_r-g_w)$ is same from the high and order unity values of the Prandtl number. In fact the recovery enthalpy g_r are governed by different expansions. However, it is fortuitous that if dissipation is neglected ($g_r = 1$), the results (4.56, 4.60) for $\sigma \approx O(1)$ expanded binomially for large σ gives the results same as (4.26, 4.36) for large σ. If the dissipation is neglected the results (4.56, 4.60) reduces to those by Merk (1959) and Evans (1967).

Let us now study our result for the recovery factor (4.24) and (4.24) for high and low Prandtl number. Again convergence of both the series becomes poor as β decreases. We have again improved the convergence of low σ result. The comparison of present results with various availables results (only available

for $\beta = 0$) is made in Fig. 4.6. Our result (4.25) for high σ gives $r_f = 1.048$ at $\sigma = 1$. If the first term of the result due to Meksyn (1961) is used, it overestimates r_f by 92% and if first two terms due to Narasimha and Vasantha (1966) is used there is an underestimate of r_f by 62% at $\sigma = 1$. The Eulerised results from low σ series show that at $\sigma = 1$, $r_f = 1.0019$ (the exact result of Pohlhausen 1921 is unity). If the first term in low σ series is used as given by Stewartson (1964), There is an underestimate r_f of 5% near $\sigma = 0.25$. If two terms of Morgan et.al (1958) are plotted there is an overestimate of r_f about 6% near $\sigma = 0.25$. It is interesting again to see from Fig. 4.6 that the results from low and high σ series fills the gap in the intermediate range. Similar comments apply to the results with pressure gradients. Lastly, our results (4.55) for Prandtl number of order unity has a graphical representation similar to that shown in Fig. 3.7.

From the above study, it is now clear that how the simple study presented here for two limiting cases: low and high values of Prandtl number, fills the gap in the intermediate region where Prandtl number is of order unity, provided a sufficiently large number of terms in each of the expansions (low and high σ) is computed and the series are properly Eulerised. Thus, in some series, it is enough only to study the two limiting cases.

CHAPTER 5

SECOND ORDER EFFECTS ON LAMINAR BOUNDARY LAYER IN NON SIMILAR FLOW OF A CONSTANT PROPERTY FLUID

5.1 Introduction:

The Prandtl boundary layer theory provides the leading term in an asymptotic expansion for large Reynolds number R, and does not account for what Rott and Lenard (1959) call the secondary effects. These may be classified as those due to transverse curvature, longitudinal curvature, vorticity gradient, stagnation enthalpy gradient, velocity slip, temperature jump and displacement. These effects are of order $R^{-1/2}$ compared to those of classical boundary layer theory and ~~becomes~~ become significant where R is only moderately large. Van Dyke (1962a) gave a systematic method of improving the classical boundary layer theory for these effects of order $R^{-1/2}$ and is described in Chapter 2. A critical comparison of this theory with various others is given in Chapter 1.

Van Dyke (1962b) has applied the theory so formulated to study the special cases of plane flow, axisymmetric flow and of cusped leading edge. He studied the leading terms in the Blasius series expansion in the coordinate along the body. However, at present no general solution of the second order boundary layer theory even for similar flows is available. The purpose of this study is to provide a general

analysis of the Gortler (1957) type for non similar flows with heat transfer. In the spirit of what has been said in Chapters 1 and 2, we shall be studying the second order effects of transverse curvature, longitudinal curvature, vorticity gradient, total enthalpy gradient and displacement in plane and axisymmetric flows of a constant property fluid.

5.2 Governing Equations:

The governing equations for second order boundary layer theory are obtained from Navier-Stokes equations by the method of matched asymptotic expansions. This method results in replacing the Navier-Stokes equations by two separate sets of equations, one set is valid in the inner region of order $R^{-1/2}$ near the wall and other outside this region of order $R^{-1/2}$. The solutions of these two regions are matched in an overlap domain. For the present study of constant property fluid flows with small temperature changes, we can neglect dissipation terms in the energy equation (this being easily justified). Now the governing equations of motion for the first and second order boundary layer theory as given in Chapter 2 are as follows.

First order boundary layer equations:

continuity

$$(r^j u_1)_s + (r^j v_1)_N = 0 \tag{5.1a}$$

momentum

$$L_1(u_1) - u_{1NN} = U_1 U_{1s} \tag{5.1b}$$

energy

$$L_1(h_1) - \sigma^{-1} h_{1NN} = 0 \qquad (5.1c)$$

boundary conditions

$$u_1(s, 0) = 0 = v_1(s, 0), \; h_1(s, 0) = h_w \qquad (5.1d)$$

matching conditions

$$\left.\begin{array}{l} u_1(s, N) \sim U_1(s, 0) \equiv U_1 \\ h_1(s, N) \sim S_1(0) \equiv S_1 \, . \end{array}\right] \text{as } N \to \infty \qquad (5.1e)$$

The second order boundary layer equations:

continuity

$$(r^j u_2)_s + (r^j v_2)_N = - Kr^j (Nv_1)_N$$
$$- j(r^{j-1} \cos\theta \, Nu_1)_s - (r^{j-1} \cos\theta \, Nv_1)_N \qquad (5.2a)$$

momentum

$$L_1(u_2) + L_2(u_1) - u_{2NN} = (j \cos\theta/r) \, u_{1N}$$
$$+ K(Nu_1 u_{1s} - NU_1 U_{1s} + u_{1N} - u_1 v_1)$$
$$+ (KU_1^2 + K \int_N^\infty U_1^2 - u_1^2 \, (s, N) \, dn)_s$$
$$+ r^j B_1'(0) \, V_2(S,0) + (U_1 \, U_2)_s \qquad (5.2b)$$

energy

$$L_1(h_2) + L_2(h_1) - \sigma^{-1} h_{2NN} = (j \cos\theta/r) \, h_{1N} \sigma^{-1}$$
$$+ K(\sigma^{-1} h_{1N} + Nu_1 h_{1s}) \qquad (5.2c)$$

boundary conditions

$$u_2(s,0) = 0 = v_2(s, 0) \, , \; h_2(s, 0)=0 \qquad (5.2d)$$

matching conditions

$$\left.\begin{array}{l} u_2(s,N) \sim -KNU_1 + r^j NB_1'(0) + U_2 \\ h_2(s,N) \sim \Psi_1(s, N) \, S_1'(0) \end{array}\right] \text{as } N \to \infty \qquad (5.2e)$$

,

where

$$V_2(s, 0) = \lim_{N \to \infty} (v_1 - Nv_{1N}),$$

$$L_r(\text{operator}) = u_r\, \partial/\partial s + v_r\, \partial/\partial N$$

and

$$U_1 = U_1(s,0) \text{ and } U_2 = U_2(s, 0).$$

5.3 Boundary Layer Characteristics:

To study the detailed behaviour of the boundary layer it is also essential to study the second order effects on various boundary layer characteristics such as, the shear stress and heat transfer at the wall, the displacement and momentum thicknesses; these quantities are defined as follows:

shear stress:

$$\begin{aligned} \tau &= R^{-1/2}\, u_N\, (s, 0) \\ &= R^{-1/2}\, \tau_1 + R^{-1}\tau_2 + \cdots \end{aligned} \tag{5.3}$$

heat transfer:

$$\begin{aligned} q &= -\sigma^{-1}\, R^{-1/2}\, h_N(s, 0) \\ &= R^{-1/2}\, q_1 + R^{-1}q_2 + \cdots \end{aligned} \tag{5.4}$$

displacement thickness:

$$\begin{aligned} \delta^* &= R^{-1/2} \int_0^\infty \left[u(s, \infty) - u(s, N)\right] /U_1\, dN \\ &= R^{-1/2}\, \delta_1^* + R^{-1}\, \delta_2^* + \cdots \end{aligned} \tag{5.5}$$

momentum thickness:

$$\begin{aligned} \Delta &= R^{-1/2} \int_0^\infty \left[u(s, \infty)-u(s,N)\right] u(s,N)/U_1^2\, dN \\ &= R^{-1/2}\, \Delta_1 + R^{-1}\Delta_2 + \cdots. \end{aligned} \tag{5.6}$$

Now, the substitution, of the inner expansions (2.17) in the above and the collection the coefficients of various powers of R give the boundary layer characteristics for successive approximations. Thus, the leading terms gives the characteristics for the first order boundary layer as

$$\begin{aligned}
\tau_1 &= u_{1N}(s,\ 0)\\
q_1 &= -\sigma^{-1}\ h_{1N}(s,\ 0)\\
\delta_1^* &= \int_0^\infty (U_1-u_1)U_1^{-1}\ dN\\
\Delta_1 &= \int_0^\infty (U_1-u_1)u_1/U_1\ dN.
\end{aligned}$$

The next higher terms give the characteristics for the second order boundary layer as

$$\begin{aligned}
\tau_2 &= u_{2N}(s,\ 0)\\
q_2 &= -\sigma^{-1}\ h_{2N}(s,0)\\
\delta_2^* &= \int_0^\infty (U_2 - KNU_1 + r^jB_1'(0)N - u_2)/U_1\ dN \qquad (5.8)\\
\Delta_2 &= \int_0^\infty \Big[(U_2 - KNU_1 + r^jB_1'(0)N-u_2)u_1/U_1^2\\
&\quad + (U_1 - u_1)u_2/U_1^2\Big]dN\ .
\end{aligned}$$

The various boundary layer parameters τ_1, τ_2, ... Δ_1, Δ_2 are all of order unity.

5.4 Choice of Independent Variables:

In order to study the boundary layer flows one generally resorts to similarity solutions (i.e., the partial differential equations are reduced to total equations). However, in majority of the engineering problems with diverse applications, it is a rare occurance indeed when all the conditions for the similarity are satisfied. In literature various methods have been used to study the non similar flows, a few of them are mentioned below.

In the first of these methods instead of solving the boundary layer equations, we adopt approximate techniques such as integral methods etc., see, Devan 1965 and Devan and Oberai 1964 (the references in this para are of those works which pertain specifically to the second order boundary layer). A second type of non similar calculation employs a strictly numerical approach. In this method the full partial differential equations of the problem are solved numerically (see, Davis and Flugge-Lotz 1964b and Fannelop and Flugge-Lotz 1966). There are various difficulties with this type of numerical solution. First is that a new calculation is to be performed for each set of data in a specific problem and one cannot draw general conclusions about the behaviour of the solution with respect to parameters. Secondly, the numerical methods

are expensive. The third method is to use some sort of local similarity. In this class we can mention the method of Blasius (1908), Görtler (1957) and that of Meksyn (1961) which was later developed by Merk (1959). Bush (1964) later incorporated the important terms of corrections, neglected by Merk (1959). This later method studied by Meksyn, Merk and Bush will be cited below as MMB.

The main shortcoming of Blasius series method is well known, namely its poor convergence for slender body shapes. The reason for this is as follows. In Blasius series method the outer boundary condition (i.e., the $U_1(s)$ approaches asymptotically at large distances from the wall) is satisfied by expanding $U_1(s)$ in a power series of s and adjusting each term of Blasius series so as to assume the value of the corresponding term of the outer velocity series asymptotically. However, if $U_1(s)$ does not happen to a polynomial of low degree (as is the case in slender body flow), the outer boundary condition is satisfied with sufficient accuracy only if an appropriate large number of terms of Blasius series are taken, and the rapidity of convergence of Blasius series will depend to a considerable extent on the convergence of the power series of $U_1(s)$.

The later methods of Görtler and MMB have the advantage that the leading terms of their series satisfy the outer

boundary condition exactly at all cross-sections along the wall. Therefore all further terms of the series give corrections only to the inner part of the boundary layer. Secondly, the well known similarity solutions forms the leading terms in the above method. But in these methods the leading term itself, gives a good approximation for the boundary layer flow from origin to down stream crosssections surprisingly far away. Finally, the above methods of Görtler and MMB are not only applicable to all cases of cylinders with round noses (cases to which Blasius series also applies) but may also be applied to cylinders with forward apex line. However, very near to the point of zero skin friction none of the above method gives satisfactory results. Nevertheless, these methods can be used to calculate the flows ahead of separation. The calculations can later be continued with the aid of suitable numerical technique.

The underlying principle in these methods of Görtler and MMB is the following, 'if the boundary layer is at all times very nearly described by a similar solution, then the direct effect of the non similar terms can be calculated by expanding the full boundary layer equations in terms of a small parameter which measure the departure of the solution from the similar one'. The main difference between Görtler and MMB methods is in the choice of the expansion parameters. In

Görtler method the expansion parameter is a variable defined by

$$\xi = \int_0^s U_1 \, ds$$

and in MMB method the expansion parameter is $d\beta/d\chi$ where

$$\beta = 2d \ln U_1/d \ln\xi \ , \quad \chi = \ln \sqrt{\xi} \ .$$

No comparison of the above two methods is available in the literature. The numerical solution of the resulting equations in MMB method is relatively much more involved due to the fact that the non-similar equations contain terms of the type $\partial f_r/\partial\beta$ and $\partial^2 f_r/\partial\beta\,\partial\chi$, $r = 0,1,2\ldots$. For example even to obtain the first non similar term we need sufficiently accurate values of quantities like $\partial f_0/\partial\beta$, $\partial^2 f_0/\partial\beta, \partial\chi$ (f_0 being the similarity solution) and the procedure for obtaining the successive terms will be much more involved.

In view of the above we adopt the Görtler (1957) type method for our study of the second order boundary layer theory. However, the Görtler (1957) variables (ξ, η) are designed for two dimensional flow. In Schlichting (1962), it is mentioned that V. Saljnikov has extended the Gortler scheme to axisymmetric case for first order boundary layer. The work of V. Saljnikov is not available to the present author. However, it is fairly obvious that the following scheme (a combination of Görtler and Mangler transformations) works for both the

two-dimensional ($j = 0$) and the axisymmetric ($j = 1$) cases. We pose

$$\xi = \int_0^s r^{2j}\, U_1\, ds, \quad \eta = NU_1 r^j/(2\xi)^{1/2}. \qquad (5.9)$$

5.5 <u>First Order Boundary Layer</u>:

The equation of continuity (5.1a) for the first order boundary layer is satisfied by introducing the first order stream function ψ_1, i.e.,

$$\psi_{1N} = r^j u_1, \qquad \psi_{1s} = -r^j v_1. \qquad (5.10)$$

The stream function ψ_1 and the enthalpy h_1 are now expressed in terms of our variables ξ, η defined by (5.9) in the form

$$\psi_1(s, N) = \sqrt{(2\xi)}\, f(\xi, \eta) \qquad (5.11a)$$

$$h_1\,(s, N) = (h_w - S_1)\, g(\xi, \eta) + S_1. \qquad (5.11b)$$

On the basis of (5.10) and (5.11) the velocity components are

$$u_1 = U_1 f' \qquad (5.12)$$

$$v_1 = -U_1 r^j \left[f + 2\xi f_\xi + (\beta + j\Lambda_s - 1)\eta f' \right]/\sqrt{(2\xi)} \qquad (5.12)$$

where

$$\Lambda_s = (2\xi/r)\,(dr/d\xi)$$

$$= (2dr/ds)/(U_1 r^{2j}) \int_0^s U_1 r^{2j}\, ds.$$

The first order boundary layer equations (5.1) take the form

momentum

$$f''' + ff'' + \beta(1 - f'^2) = 2\xi(f' f'_\xi - f_\xi\, f'') \qquad (5.13)$$

energy

$$\sigma^{-1} g'' + fg' - \Lambda f' g = 2\xi(f' g_\xi - f_\xi\, g') \qquad (5.14)$$

boundary conditions

$$f(\xi, 0) + 2\xi f_{\xi}(\xi, 0) = 0$$
$$f'(\xi, 0) = 0, \quad f'(\xi, \infty) = 1 \tag{5.15}$$
$$g(\xi, 0) = 1, \quad g(\xi, \infty) = 0. \tag{5.16}$$

Here the dashes indicate the partial differentiation with respect to η.

The quantity β, defined by

$$\beta(\xi) = 2d \ln U_1 / d \ln \xi$$
$$= (2U_{1s}/U_1^2 r^{2j}) \int_0^s U_1 r^{2j} ds \tag{5.17}$$

is called the principal function by Gortler. This principal function is of fundamental importance in the structure of the solution. Thus, only the form of $\beta(\xi)$ differs from case to case in the boundary layer problems (5.13) and (5.15). It is easily shown (see, Gortler 1957) that if two problems are similar in the sense of Reynolds similarity they have same principal function. Further, the behaviour of the solution at singular point (say $s = 0$) depends on the behaviour of the principal function at $\xi = 0$. For the special case of two dimensional flow with $U_1 \alpha s^m$, the principal function β reduces to the Hartree parameter of wedge flows. This principal function β is as meaningful for a non similar flow as for a similar flow, the main difference in two applications being that while β remains constant in the stream direction when flow is similar it varies when the flow is not similar.

The principal function $\beta(\xi)$ depends not only on the local values of the outer velocity U_1 and outer velocity gradient U_{1s} but also on the history of the flow upstream from the cross-section $\int_0^s U_1 r^{2j} ds$. The method, of course, requires the knowledge of the distribution of the main stream velocity outside the boundary layer. This may be interms of a formula fitted to an experimental data or to obtain theoretically by solving the outer flow.

The quantity $\Lambda(\xi)$ defined by

$$\Lambda(\xi) = 2\xi(h_w - S_1)^{-1} dh_w/d\xi$$
$$= \frac{2(h_w - S_1)^{-1}}{U_1 r^{2j}} \frac{dh_w}{ds} \int_0^s U_1 r^{2j} ds \qquad (5.18)$$

will be called the principal thermal function. It is interesting to note that Λ arises in the temperature problems in a manner similar to that in which β arises in the velocity problem. Further, comments similar to those mentioned in previous para for β apply to Λ. In our present study we shall be study the case when wall temperature is prescribed to a constant value.

We choose to study the situation where the velocity $U_1(s, 0)$ may be represented by the convergent power series in the interval $0 \leq s \leq s_1$ (where $s = s_1$ denotes any suitable cross section not further down stream than that of a possible separation point of the boundary layer)*.

$$U_1(s,\ 0) = x^m \sum_{m=0} \lambda_{1n}\ x^{n(m+1)},\ -1<m<\infty \tag{5.19}$$

where $x = \int_0^s r^{2j}\ ds.$

Further, the relationship between ξ and x is

$$\xi = \frac{x^{m+1}}{m+1} \sum_{r=0} \frac{\lambda_{1r}}{r+1}\ x^{r(m+1)}. \tag{5.20}$$

The form of the principal function β corresponding to (5.19) is

$$\beta = \sum_{i=0} \beta_i\ \xi^i \tag{5.21}$$

where $\beta_o = 2m/(m+1)$ and β_i depends on coefficients λ_{1n} of equation (5.19).

Now to study the first order boundary layer problem (5.13) to (5.16) we assume that f and g are represented by the following power series

$$\begin{aligned} f(\xi,\eta) &= f_o(\eta) + \xi f^{(1)}(\eta) + \xi^2 f^{(2)}(\eta) + \ldots \\ g(\xi,\eta) &= g_o(\eta) + \xi g^{(1)}(\eta) + \xi^2 g^{(2)}(\eta) + \ldots \end{aligned} \tag{5.22}$$

which converge in the interval $0 \leq \xi \leq \xi_1$. Substituting (5.21), (5.22) in equations (5.13) to (5.16) and collecting the coefficients of the like powers of ξ we obtain the equations for successive approximations. Thus the terms independent of ξ are

$$\begin{aligned} & f_o''' + f_o f_o'' + \beta_o(1-f_o'^2) = 0 \\ & \sigma^{-1} g_o'' + f_o g_o' = 0 \\ & f_o(0) = 0 = f_o'(0), \qquad f_o'(\infty) = 1 \\ & g_o(0) = 1, \qquad g_o(\infty) = 0. \end{aligned} \tag{5.23}$$

The terms of order ξ are

$$
\begin{aligned}
Y_o(f^{(1)}) &= \beta_1 \,(1-f_o'^2) \\
Z_o(g^{(1)}) &= -3f^{(1)}g_o' \\
f^{(1)}(0) &= f'^{(1)}(0) = f'^{(1)}(\infty) = 0 \\
g^{(1)}(0) &= 0 = g^{(1)}(\infty).
\end{aligned}
\tag{5.24}
$$

The terms of order ξ^2 are

$$
\begin{aligned}
Y_1(f^{(2)}) &= \beta_2(f_o'^2 - 1) + \beta_1^2(2f_o'f'^{(1)} - 3f^{(1)}f''^{(1)} \\
&\qquad + (\beta_o + 2)f'^{(1)2} \quad) \\
Z_1(g^{(2)}) &= 2f'^{(1)}g^{(1)} - 3f^{(1)}g'^{(1)} - 5f^{(2)}g_o' \\
f^{(2)}(0) &= f'^{(2)}(0) = f'^{(2)}(\infty) = 0 \\
g^{(2)}(0) &= 0 = g^{(2)}(\infty).
\end{aligned}
\tag{5.25}
$$

Here

$$
\begin{aligned}
Y_r(f) &= f''' + f_of'' - 2(\beta_o+r)f_o'f' + (1+2r)f_o''f \\
Z_r(g) &= \sigma^{-1}g'' + f_og' - 2rf_o'g\,.
\end{aligned}
\tag{5.26}
$$

It is to be noted that equations (5.23) governing the functions f_o and g_o are the same as those of an equivalent wedge of angle $\beta_o\pi$. Further, the independent variable η on which f_o depends reduces to the variable of similar solutions when $\beta(\xi) = \text{const.}$ $(U_1 \propto x^m)$. In general case our variable η approaches the variable of similar flows for sufficiently small x or ξ . It can therefore be expected that the first order approximation f_o will be good not only in the immediate viscinity of the origin but even rather far

downstream. Moreover, the functions f_o and g_o satisfy the outer boundary conditions exactly, thus all further terms f_m etc represent only the correction to inner part of the boundary layer.

To solve the equations f_o, g_o we need to know β_o, while for $f^{(i)}$, $g^{(i)}$ we need β_o, β_1 - - - - β_i. Thus for each set of data of a specific problem we have to solve the entire set of equations. However, we can take the advantage of the linearity of the higher order equations by rephrasing our problem in terms of some functions which are independent of the explicit data of a particular problem. Thus, for any fixed value of β_o, it is possible to reduce the equations $f^{(i)}(\eta, \beta_o, \beta_1, \ldots \beta_i)$ to a linear combinations of functions not depending on β_1, β_2, ... β_i and therefore for any fixed value of β_o the universal functions can be tabulated once and for all. The stated reduction is achieved by setting for a fixed value of β_o

$$\begin{aligned} f^{(1)} &= \beta_1 f_1 \\ f^{(2)} &= \beta_1^2 f_{11} + \beta_2 f_2. \end{aligned} \tag{5.27}$$

Similarly for fixed value of β_o and σ the corresponding energy equation may be splitted into universal functions by setting

$$\begin{aligned} g^{(1)} &= \beta_1 g_1 \\ g^{(2)} &= \beta_1^2\, g_{11} + \beta_2 g_2. \end{aligned} \tag{5.27a}$$

Now the final form of the expansions (5.22) which leads to desired universal functions are

$$f = f_o + \xi\, \beta_1 f_1 + \xi^2(\beta_1^2 f_{11} + \beta_2 f_2) + \cdots$$
$$g = g_o + \xi\, \beta_1 g_1 + \xi^2(\beta_1^2 g_{11} + \beta_2 g_2) + \cdots. \qquad (5.28)$$

Thus, the final form for the terms of order ξ is

$$\begin{aligned} Y_o(f_1) &= f_o'^2 - 1 \\ Z_o(g_1) &= -\,3f_1 g_o' \\ f_1(0) &= f_1'(0) = f_1'(\infty) = 0 \\ g_1(0) &= 0 = g_1(\infty), \end{aligned} \qquad (5.29)$$

and the terms of order ξ^2 give the following two set of equations

$$\begin{aligned} Y_1(f_2) &= f_o'^2 - 1 \\ Z_1(g_2) &= -5f_2 g_o' \\ f_2(0) &= f_2'(0) = f_2'(\infty) = 0 \\ g_2(0) &= g_2'(\infty) = 0, \end{aligned} \qquad (5.30)$$

and
$$\begin{aligned} Y_1(f_{11}) &= 2f_o' f_1' - 3f_1 f_1'' + (\beta_o + 2)\, f_1'^2 \\ Z_1(g_{11}) &= 2f_1' g_1 - 3f_1 g_1' - 5f_{11} g_o' \\ f_{11}(0) &= f_{11}'(0) = f_{11}'(\infty) = 0 \\ g_{11}(0) &= g_{11}(\infty) = 0. \end{aligned} \qquad (5.31)$$

Now the above set of equations (5.29), (5.30), (5.31) etc. are independent of β_1, β_2 ... and can be solved once and for all for fixed values of β_o and σ. The results may be tabulated for the universal functions. Knowing β_1, β_2 ... in specific problems. The results may be obtained immediately

from the tabulated universal functions.

Further, our equations (5.29), (5.30), (5.31) etc. obtained for the plane and the axisymmetric flows are of the same form as those of Görtler (1957) for plane flows. Görtler (1957) studied the nonsimilar momentum equations in (5.23), (5.24), (5.25) etc. for $\beta_0 = 0$, and 1 only. Thus, the validity of the Görtler series for negative β_0 is not known. Sparrow (1958) studied the energy equations of the two-dimensional nonsimilar flows for $\beta = 1$, and $\sigma = 1$. Hence, even the study of the above equations (5.29) to (5.31) is not exhaustive in the literature. In the present work, in addition to the study of the second order boundary layer problem some of the nonsimilar equations for the first order will also be studied in detail.

The first equation in (5.23) for the first term of the first order boundary layer theory, is nonlinear and is well known Falkner-Skan equation, whose solution is studied by various authors. Some general remarks about the existence, uniqueness of the solutions of this equation has already been made in Chapter 4. In the present work the initial value $f_0''(0)$ is taken from the tables of Smith (1954) and solution is generated at a step size of 0.05. This solution is shown in Fig. 5.1. Now the energy equation in (5.23) and the equations (5.24) are linear and constitute the two point boundary value problems. The solution to above mentioned equations are obtained by integrating them (the details of procedure are given in Appendix C) by Runge-Kutta-Gill method

on IBM 7044 Computer at IIT Kanpur. The solutions are obtained for both the accelerated and decelerated flow in the range $-.198838 < \beta_o \leq 2$ and the Prandtl number of the fluid in the range 0.7 to 3. The results of the numerical solutions are displayed in Figs. 5.2 to 5.4.

First order boundary layer characteristics:

Substituting our variables (5.9) and (5.11) into the first order boundary layer characteristics (5.18) we get

$$\tau_1 \sqrt{2}/(U_1^2 r^j) = \xi^{-1/2} f''(\xi, 0)$$

$$= \xi^{-1/2} \tau_{1,o} + \beta_1 \xi^{1/2} \tau_{1,1} + \cdots \qquad (5.31)$$

$$\frac{\sqrt{2} q_1}{(S_1 - h_w)\sigma^{-1} U_1 r^j} = \xi^{-1/2} g'(\xi, 0)$$

$$= \xi^{-1/2} q_{1,o} + \beta_1 \xi^{1/2} q_{1,1} + \cdots \qquad (5.32)$$

$$\delta_1^* U_1 r^j/\sqrt{(2\xi)} = \int_0^\infty (1-f')d\eta$$

$$= \delta_{1,o}^* + \beta_1 \xi \delta_{1,1}^* + \cdots \qquad (5.33)$$

$$\Delta_1 U_1 r^j/\sqrt{(2\xi)} = \int_0^\infty (1-f')f' d\eta$$

$$= \Delta_{1,o} + \xi\beta_1 \Delta_{1,1} + \cdots. \qquad (5.34)$$

Now, substituting the variables (5.23) and collecting the coefficients of the like powers of ξ, the leading terms are

$$\tau_{1,o} = f_o''(0)$$

$$q_{1,o} = g_1'(0) \qquad (5.35)$$

$$\delta^*_{1,o} = \int_0^\infty (1-f_o') \, d\eta$$
$$\Delta_{1,o} = \int_0^\infty (1-f_o') \, f_o' \, d\eta$$

while the next higher order terms are

$$\begin{aligned} \tau_{1,1} &= f_1''(0) \\ q_{1,1} &= g_1'(0) \\ \delta^*_{1,1} &= \int_0^\infty - f_1' \, d\eta = -f_1(\infty) \\ \Delta_{1,1} &= \int_0^\infty f_1'(1-2f_o') \, d\eta \; . \end{aligned} \qquad (5.36)$$

These characteristics are also evaluated for various values of β_o and σ and are displayed in Fig. 5.5 and Fig. 5.6.

5.6 Second Order Boundary Layer Theory:

A feature of the second order boundary layer equations (5.2) —— as a matter of fact, of the second and higher order terms in all perturbation schemes —— are linear and hence, allow superposition. Following, the suggestion of Rott and Lenard (1959) we can divide these higher order problem (5.2) into a number of simpler problems, each of which has a clear physical interpretation and then superpose. However, this division is some what arbitrary and it is possible to divide these second order effects in many ways (particularly those of vorticity and displacement). Van Dyke (1962c) and Cheng (1966) have discussed how various authors (mentioned in Chapter 1) have subdivided the second order terms. The way in which these effects are divided here, in the present study, is

the same as those of Van Dyke (1962a). The terms on the right hand side of equations (5.2) fall into the following categories. In momentum equation (5.2a) the terms proportional to $(j \cos \theta/r)$ arise from the transverse curvature, K from the longitudinal curvature, $B_1'(0)$ represent the effect of vorticity in the basic on coming stream and $U_2(s, 0)$ arise from the displacement speed, i.e., the change induced in the speed at the outer edge of classical boundary layer. Further, in the matching condition (5.2e) for the energy equation (5.2c), the term proportional to $S_1'(0)$ represents the effect of stagnation enthalpy gradient in the on-coming stream.

Thus, the second order problem (5.2) can be decomposed the into five constituents by writting

$$u_2 = u_2^{(t)}+u_2^{(l)}+B_1'(0)u_2^{(v)}+S_1'(0)u_2^{(s)}+u_2^{(d)} \tag{5.37}$$

$$v_2 = v_2^{(t)}+v_2^{(l)}+B_1'(0)v_2^{(v)}+S_1'(0)v_2^{(s)}+v_2^{(d)} \tag{5.38}$$

$$h_2 = h_2^{(t)}+h_2^{(l)}+B_1'(0)h_2^{(v)}+S_1'(0)h_2^{(s)}+h_2^{(d)}. \tag{5.39}$$

Here, the superscripts t, l, v, s and d identify respectively the contributions of the transverse curvature, the longitudinal curvature, external vorticity gradient, stagnation enthalpy gradient and the displacement. We now proceed to study each of these effects separately by applying the Görtler type method.

5.6.1 Transverse Curvature Problem:

Keeping only such nonhomogenous terms in (5.2) as involve $(j \cos \theta/r)$ gives, the problem for $u_2^{(t)}$, $v_2^{(t)}$ and $h_2^{(t)}$, (now

onwards these superscripts will be dropped in this section, however, the superscripts will be used out side this section wherever necessary) is governed by

$$\left[r^j(u_2+jNu_1 \cos\theta/r)\right]_s + \left[r^j(v_2+jNv_1 \cos\theta/r)\right]_N = 0$$
$$L_1(u_2)+L_2(u_1)-u_{2NN} = ju_{1N} \cos\theta/r$$
$$L_1(h_2)+L_2(h_1)-\sigma^{-1} h_{2NN} = jh_{1N} \cos\theta/r \qquad (5.40)$$
$$u_2(s,\ 0) = v_2(s,\ 0) = u_2(s,\ \infty) = 0$$
$$h_2(s,\ 0) = 0 = h_2(s,\ \infty).$$

The equation of continuity (5.40a) is satisfied by introducing the second order stream function ψ_2, defined by

$$\psi_{2N} = r^j(u_2+jNu_1 \cos\theta/r)$$
$$\psi_{2s} = -r^j(v_2+jNv_1 \cos\theta/r). \qquad (5.41)$$

Now, the stream function ψ_2 and the enthalpy h_2 are assumed to be of the form

$$\psi_2(s,\ N) = \sqrt{(2\xi)}\ F(\xi,\eta)$$
$$h_2(s,\ N) = (h_w-S_1)G(\xi,\eta). \qquad (5.42)$$

To be consistent with the study of Sec. 5.5 (for the first order boundary layer) the relation (5.11) are used for ψ_1 and h_1. Thus, the velocity components on the basis of (5.12), (5.41) and (5.42) are

$$u_2 = U_1F' - jf'N(2\xi)^{1/2} \cos\theta/r^2 \qquad (5.43a)$$

$$v_2 = U_1 r^j \left[F+2\xi F_\xi +(\beta+j\Lambda_s-1)\eta F'\right] /(2\xi)^{1/2}$$

$$+ \left[f+2\xi f_\xi +(\beta+j\Lambda_s-1)\eta f'\right](j \cos\theta/r) \qquad (5.43b)$$

where $\Lambda_s = (2\xi/r)\ (d\xi/dr)$ will be termed as the principal body shape function. Making use of (5.10), (5.11), (5.41), (5.42) the second order problem (5.40) reduce to

$$F''' + fF'' - 2\beta f'F' + f''F + 2\xi(f_\xi F'' - F'f'_\xi - f'F'_\xi + f''F_\xi)$$

$$= k_t\left[(1+\eta f)f'' + ff' - \beta\eta(1+f'^2) + 2\xi f'f_\xi\right.$$

$$\left. + 4\eta\xi(f_\xi f'' - f'f'_\xi) - \Lambda_t\eta f'^2\right] \qquad (5.44)$$

$$\sigma^{-1}G'' + fG' - \Lambda f'G + 2\xi(f_\xi G' - f'G_\xi) = -Fg' + \Lambda F'g + 2\xi(f_\xi G' - f'G_\xi)$$

$$+ k_t\left[(-\sigma^{-1}+\eta f)g' - 2\xi\eta(f'g_\xi - f_\xi g')\right] \qquad (5.45)$$

$$F(\xi, 0) + 2\xi F_\xi(\xi, 0) = 0$$

$$F'(\xi, 0) = 0; \quad F'(\xi, \eta) = k_t\eta, \quad \eta \to \infty. \qquad (5.46)$$

$$G(\xi, 0) = 0 = G(\xi, \infty). \qquad (5.47)$$

Here $k_t(\xi)$ is the transverse curvature function defined by

$$k_t(\xi) = (2\xi)^{1/2} \cos\theta/(U_1 r^{j+1}) \qquad (5.48)$$

and the function

$$\Lambda_t(\xi) = (2\xi/k_t)\ (dk_t/d\xi) \qquad (5.49)$$

will be called the principal transverse curvature function. It is interesting to see that Λ_t arises in the transverse curvature problem in the same manner β arises in classical boundary layer equations.

In order to study the above problem in detail, we assume (in addition to (5.23) for the first order boundary layer theory) the following expansions for F, G and k_t

$$
\begin{aligned}
F(\xi,\eta) &= F^{(0)}(\eta) + \xi F^{(1)}(\eta) + \cdots \\
G(\xi,\eta) &= G^{(0)}(\eta) + \xi G^{(1)}(\eta) + \cdots \\
k_t(\xi) &= k_{ot} + \xi k_t^{(1)} + \cdots .
\end{aligned} \tag{5.50}
$$

Substituting (5.50), (5.23) in (5.44) to (5.47) and collecting the coefficient of various powers of ξ, the terms independent of ξ are

$$
\begin{aligned}
Y_0(F^{(0)}) &= k_{ot}\left[(1+\eta f_o)f_o''+f_of_o'-\eta\beta(1+f_o'^2)\right] \\
Z_0(G^{(0)}) &= -F^{(0)}g_o' + k_{ot}(-\sigma^{-1}+\eta f_o)g_o' \\
F^{(0)}(0) &= 0 = F'^{(0)}(0), \quad F'^{(0)}(\eta) = k_{ot}\eta, \ \eta\to\infty \\
G^{(0)}(0) &= 0 = G^{(0)}(\infty).
\end{aligned} \tag{5.51}
$$

and of order ξ are

$$
\begin{aligned}
Y_1(F^{(1)}) &= -3f^{(1)}F''^{(0)}+2(1+\beta_o)f'^{(1)}F'^{(0)}-f''^{(1)}F^{(0)}+2\beta_1 f_o'F'^{(0)} \\
&\quad +k_{ot}\left[(1+\eta f^{(0)})f''^{(1)}+5\eta f^{(1)}f_o''+f_of'^{(1)}+3f_o'f^{(1)}\right. \\
&\quad \left.-2(2+\beta_o)\eta f_o'f'^{(1)}+\beta_1\eta(1+f_o'^2)\right]+k_t^{(1)}\left[f_o''(1+\eta f_o)\right. \\
&\quad \left.+f_of_o' - \beta_o\eta(1+f_o'^2)-2\eta f_o'^2\right] \\
Z_1(G^{(1)}) &= -3F^{(1)}g_o'-3f^{(1)}G'^{(0)}-F^{(0)}g'^{(1)}+2F'^{(0)}g^{(1)} \\
&\quad +k_{ot}\left[-\sigma^{-1}g'^{(1)}+\eta f_og'^{(1)}-2\eta f_o'g^{(1)}+3\eta f^{(1)}g_o'\right] \\
&\quad +k_t^{(1)}(-\sigma^{-1}+\eta f_o)g_o'
\end{aligned} \tag{5.52}
$$

$$F^{(1)}(0) = 0 = F'^{(1)}(0), \qquad F'^{(1)}(\eta) = k_t^{(1)}\eta, \eta \to \infty$$
$$G^{(1)}(0) = 0 = G^{(1)}(\infty).$$

Now the above system of equations involve, in addition to β_1, β_2, β_3 ... the quantities k_{ot}, $k_t^{(1)}$ etc. However, we can rephrase the problem by introducing the proper universal functions. For the first order problem we still have to use the relations (5.27) and (5.28); for the second order problem (5.51) and (5.52) we assume

$$\begin{aligned} F^{(o)}(\eta) &= k_{ot}\, F_o(\eta) \\ G^{(o)}(\eta) &= k_{ot}\, G_o(\eta) \\ F^{(1)}(\eta) &= k_{1t}\, F_1(\eta) + \beta_1 k_{ot}\, F_{1o}(\eta) \\ G^{(1)}(\eta) &= k_{1t}\, G_1(\eta) + \beta_1 k_{ot} G_{1o}(\eta). \end{aligned} \tag{5.53}$$

$$k_t^{(1)} = k_{1t}.$$

The final form of the expansions (5.28) for the first order boundary layer are

$$\begin{aligned} f(\xi,\eta) &= f_o(\eta) + \xi\beta_1 f_1(\eta) + \cdots \\ g(\xi,\eta) &= g_o(\eta) + \xi\beta_1 g_1(\eta) + \cdots \end{aligned} \tag{5.28}$$

and the expansion (5.50) for the second order transverse curvature problem are

$$\begin{aligned} F(\xi,\eta) &= k_{ot}\, F_o(\eta) + \xi\left[k_{1t}F_1(\eta) + \beta_1 k_{ot}\, F_{10}(\eta)\right] + \cdots \\ G(\xi,\eta) &= k_{ot}\, G_o(\eta) + \xi\left[k_{1t}G_1(\eta) + \beta_1 k_{ot}\, G_{10}(\eta)\right] + \cdots \\ k_t(\xi) &= k_{ot} + \xi\, k_{1t} + \cdots. \end{aligned} \tag{5.54}$$

Now the various universal functions are as follows. The equations, (5.51) give

$$Y_0(F_0) = (1+\eta f_0)f_0'' + f_0 f_1'' + f_0 f_1' - \eta\beta_0(1+f_0'^2) \quad (5.55a)$$

$$Z_0(G_0) = (-\sigma^{-1}+\eta f_0 - F_0)\, g_0' \quad (5.55b)$$

$$F_0(0) = 0 = F_0'(0), \quad F_0(\eta) = \eta, \quad \eta \to \infty, \quad (5.55c)$$

$$G_0(0) = 0 = G_0(\infty). \quad (5.55d)$$

The equations (5.51b) leads to two sets of the universal functions; the first is

$$Y_1(F_1) = (1+\eta f_0)\, f_0'' + f_0 f_1' - \eta\beta_0(1+f_0'^2) - 2\eta f_0'^2 \quad (5.56a)$$

$$Z_1(G_1) = (-\sigma^{-1}+\eta f_0 + 3F_1) g_0' \quad (5.56b)$$

$$F_1(0) = 0 = F_1'(0), \quad F_1'(\eta) = \eta, \quad \eta \to \infty, \quad (5.56c)$$

$$G_1(0) = G_1(\infty) = 0 \quad (5.56d)$$

and the second set is

$$\begin{aligned} Y_1(F_{10}) = {} & -3 f_1 F_0'' + 2(1+\beta_0) f_1' F_0' - f_1'' F_0 + 2 f_0' F_0' \\ & + (1+\eta f_0) f_1'' + 5\eta f_1 f_0'' + f_0 f_1' + 3 f_0' f_1 \\ & -2\eta(2+\beta_0) f_0' f_1' - \eta(1+f_0'^2) \end{aligned} \quad (5.57a)$$

$$\begin{aligned} Z_1(G_{10}) = {} & -3F_{10} g_0' - 3 f_1 G_0' - F_0 g_1' + 2F_0' g_1 \\ & -\sigma^{-1} g_1' + \eta f_0 g_1' - 2\eta f_0' g_1 + 3\eta f_i g_0' \end{aligned} \quad (5.57b)$$

$$F_{10}(0) = F_{10}'(0) = F_{10}'(\infty) = 0 \quad (5.57c)$$

$$G_{10}(0) = 0 = G_{10}(\infty). \quad (5.57d)$$

The above three sets of equations (5.55), (5.56) and (5.57) which involve the solutions of the corresponding first order boundary layer equations are integrated numerically on IBM 7044 computer at IIT Kanpur by Runge-Kutta-Gill method. The procedure of solution is disscussed in the Appendix C . The solutions are obtained for both the accelerating and the decelerating flows ($-.198838 < \beta_o \leq 2$) and for the values of the Prandtl number between 0.7 and 3.0. The results are displayed in Figs.5.7 to 5.12.

The transverse curvature effects on various boundary layer characteristics (5.8) are still to be studied. Substituting (5.9), (5.12) and (5.42) the contribution of the transverse curvature may be written as

$$\tau_2 \sqrt{(2\xi)}/(U_1^2 r^j) = F''(\xi, 0) \tag{5.58}$$

$$= k_{ot}\,\tau_{2,0} + \xi(k_{1t}\,\tau_{2,1} + \beta_1 k_{ot}\,\tau_{2,10}) + \cdots$$

$$\frac{q_2\,(2\xi)^{1/2}}{(S_1-h_w)\sigma^{-1}U_1 r^j} = G'(\xi, 0)$$

$$= q_{2,0} + \xi(k_{1t} q_{2,1} + \beta_1 k_{ot}\, q_{2,10}) + \cdots \tag{5.59}$$

$$\delta_2^* U_1 r^j/\sqrt{(2\xi)} = \int_0^\infty k_t \eta f' - F' \, d\eta$$

$$= k_{ot}\,\delta^*_{2,0} + \xi(k_{1t}\,\delta^*_{2,1} + \beta_1 k_{ot}\delta^*_{2,10}) + \cdots \tag{5.60}$$

$$\Delta_2\, U_1 r^j/\sqrt{(2\xi)} = \int_0^\infty (1-2f')(F'-k_t\,\eta f')\,d\eta$$

$$= k_{ot}\Delta_{2,0} + \xi(k_{1t}\Delta_{2,1} + \beta_1 k_{ot}\Delta_{2,10}) + \cdots \tag{5.61}$$

Now substituting the expansions (5.28a) and (5.54) in the above and collecting the coefficients of like powers of ξ , we get,

$$\begin{aligned}
\tau_{2,0} &= F_0''(0) \\
q_{2,0} &= G_0'(0) \\
\delta^*_{2,0} &= \int_0^\infty \eta f_0' - F_0' \, d\eta \\
\Delta_{2,0} &= \int_0^\infty (1-2\eta f_0') \, (F_0' - \eta f_0') \, d\eta
\end{aligned} \tag{5.62}$$

and the terms of order ξ gives two sets; the first is

$$\begin{aligned}
\tau_{2,1} &= F_1''(0) \\
q_{2,1} &= G_1'(0) \\
\delta^*_{2,1} &= \int_0^\infty \eta f_0' - F_1' \, d \\
\Delta_{2,1} &= \int_0^\infty (1-2f_0')(F_1' - \eta f_0') \, d\eta
\end{aligned} \tag{5.63}$$

and the second set is

$$\begin{aligned}
\tau_{2,10} &= F_{10}''(0) \\
q_{2,10} &= G_{10}'(0) \\
\delta^*_{2,10} &= \int_0^\infty f_1' - F_{10}' \, d\eta \\
\Delta_{2,10} &= \int_0^\infty (1-2\, f_0') \, (F_{10}' - \eta f_1') - 2f_1(F_0' - \eta f_0') \, d\eta
\end{aligned}$$

The above mentioned boundary layer characteristics (5.62), (5.63), (5.64) are also evaluated and are shown in Figs. 5 13 to 5.15.

5.6.2 Longitudinal Curvature Problem:

Now keeping only those nonhomogeneous terms in (5.2) which involve K, we obtain the following problem for $u_s^{(1)}$ etc. (Superscript 1 will again be omitted here)

$$
\begin{aligned}
&(r^j u_2)_s + (r^j v_2 + KNr^j v_1)_N = 0 \\
&L_1(u_2)+L_2(u_1)-u_{2NN} = K(Nu_1 u_{1s}-NU_1 U_{1s}+u_{1N}-u_1 v_1) \\
&L_1(h_2)+L_2(h_1)-\sigma^{-1}h_{2NN} = K(\sigma^{-1}h_{1N}+Nu_1 h_{1s}) \\
&u_2(s,\ 0) = 0 = v_2(s,\ 0),\ u_2(s,\ N) = -KN,\ N \to \infty \\
&h_2(s,\ 0) = 0 = h_2(s,\ \infty).
\end{aligned}
\tag{5.65}
$$

Here, the second order stream function takes the form

$$
\begin{aligned}
\psi_{2N} &= r^j u_2 \\
\psi_{2s} &= -r^j(v_2+KNv_1)
\end{aligned}
\tag{5.66}
$$

and the ψ_2 and h_2 are assumed to be of the form

$$
\begin{aligned}
\psi_2(s,\ N) &= \sqrt{(2\xi)}\ F(\xi,\eta) \\
h_2(s,\ N) &= (h_w-S_1)\ G(\xi,\eta).
\end{aligned}
\tag{5.67}
$$

So that, the velocity components are given by

$$
\begin{aligned}
u_2 &= U_1 F' \\
v_2 &= -U_1 r^j \left[F+2\xi F_\xi +(\beta+j\Lambda_s-1)\eta F'\right]/(2\xi)^{1/2} \\
&\quad + K\left[f+2\xi f_\xi +(\beta+j\Lambda_s-1)\eta f'\right].
\end{aligned}
\tag{5.68}
$$

The differential equations for the longitudinal curvature problem (5.65) become

$$F''' + fF'' - 2\beta f'F' + f''F + 2\xi\,(f_{\xi}\,F'' - F'f'_{\xi} \;\; - f'F'_{\xi} \;\; + f''F_{\xi})$$
$$= k_1\Big[\,(2\beta+\Lambda_1)\Big(\int_0^{\eta} f'^2 d\eta + \int_0^{\infty} 1-f'^2\, d\eta\,\Big)$$
$$+ (\eta f-1)f'' - ff' + \beta\eta(1-f'^2) + 2\xi\eta\,(f_{\xi}\,f'' \tag{5.69}$$
$$- f'f'_{\xi}\;\;) - 2\xi\,(f'f_{\xi} + \int_{\eta}^{\infty} f'f'_{\xi}\;\; d\eta\,)\Big]$$

$$\sigma^{-1}G'' + fG' - \Lambda f'G + 2\xi\,(f_{\xi}\,G' - f'G_{\xi}\,) = -Fg'$$
$$+ \Lambda F'g + 2\xi\,(F'g_{\xi} - F_{\xi}\;\; g') + k_1\Big[(-\sigma^{-1} + \eta f)g'$$
$$-2\xi\eta\,(f'g_{\xi} - f_{\xi}\,g')\Big] \tag{5.70}$$

$$F(\xi\,,\,0) + 2\xi F_{\xi}\;\;(\xi\,,\,0) = 0$$
$$F'(\xi\,,\,0) = 0,\quad F'(\xi\,,\eta) = k_1\eta\,,\quad \eta\to\infty. \tag{5.71}$$
$$G(\xi\,,\,0) = 0 = G(\xi\,,\,\infty) \tag{5.72}$$

where

$$k_1(\xi) = K(2\xi)^{1/2}/(U_1 r^j) \tag{5.73}$$

will be called the longitudinal curvature function and

$$\Lambda_1(\xi) = (2\xi/k_1)\,(dk_1/d\xi) \tag{5.74}$$

the principal longitudinal curvature function. The role of the latter function Λ_1 is similar to that of $\beta, \Lambda, \Lambda_t$ explained earlier.

It is interesting to see that for the two dimensional flows ($j = 0$) with $U_1 \propto s^m$ our $k_1(\xi)$ is constant or in other words

$$K \propto s^{\frac{m-1}{2}}$$

which represents the similarity surfaces of the studies of Murphy (1953, 1965) and Narasimha and Ojha (1967).

To study the longitudinal curvature problem we have to assume that F, G and k_1 may be represented by the series similar to the transverse curvature series (5.50) and then to split them into proper universal functions. Steps employed here are the same as in Sec. 3.6.1. So we shall be writing the final form of the expansion for F, G and k_1 which leads directly to the universal functions

$$F = k_{o1} F_o + \xi(k_{11}F_1 + \beta_1 k_{o1} F_{10}) + \cdots$$

$$G = k_{o1} G_o + \xi(k_{11}G_1 + \beta_1 k_{o1} G_{10}) + \cdots \tag{5.75}$$

where $k_1 = k_{o1} + \xi\, k_{11} + \cdots$.

Substituting the (5.75) and (5.28) in (5.69) to (5.72) and collecting the coefficients of like powers of ξ we obtain the problems for F_o, F_1 and F_{10}. The problem for F_o is

$$Y_o(F_o) = (\eta f_o - 1) f_o'' - f_o f_o' + \beta_o \eta (1 - f_o'^2) \tag{5.76a}$$

$$+ 2\beta_o (f_o'' + f_o f_o' + \beta_o \eta + \alpha_o)/(1+\beta_o)$$

$$Z_o(G_o) = (-\sigma^{-1} + \eta f_o - F_o)\, g_o' \tag{5.76b}$$

$$F_o(0) = 0 = F_o'(0), \quad F_o'(\eta) = -\eta, \ \eta \to \infty, \tag{5.76c}$$

$$G_o(0) = 0 = G_o(\infty). \tag{5.76d}$$

Here the equation (5.76a) is due to Narasimha and Ojha (1967).

The problem for F_1 is

$$Y_1(F_1) = (\eta f_o + 1) f_o'' + f_o f_o' + \eta\beta(3 - f_o'^2) + 2\alpha_o \tag{5.77a}$$

$$Z_1(G_1) = (-\sigma^{-1} + \eta f_o - 3F_1) g_o' \tag{5.77b}$$

$$F_1(0) = 0 = F_1'(0) \qquad F_1'(\eta) = -\eta,\ \eta \to \infty, \tag{5.77c}$$

$$G_1(0) = 0 = G_1(\infty). \tag{5.77d}$$

Finally the problem for F_{10} is

$$\begin{aligned} Y_1(F_{10}) = {} & -3f_1F_o''+2(1+\beta_o)f_1'F_o'-f_1''F_o+2f_o'F_o' \\ & +(\eta f_o-1)f_o''+3\eta f_1f_o''-2\eta(1+\beta_o)f_o'f_1''-3f_1f_o' \\ & -f_of_1'+\eta(1-f_o'^2)+2(f_1''+f_of_1'+3f_1f_o'+3\alpha_1)(\beta_o+1)/(\beta_o+3) \\ & +\left[4(f_o''+f_of_o'+\alpha_o)+2\eta(\beta_o^2+4\beta_o+1)\right]/(\beta_o+1)(\beta_o+3) \end{aligned} \tag{5.78a}$$

$$\begin{aligned} Z_1(G_{10}) = {} & -3f_1G_c' - 3(F_{10}-\eta f_1)g_o'+2(F_o'-\eta f_o')g_1 \\ & -(\sigma^{-1}-\eta f_o+F_o)g_1' \end{aligned} \tag{5.78b}$$

$$F_{10}(0) = F_{10}'(0) = F_{10}'(\infty) = 0 \tag{5.78c}$$

$$G_{10}(0) = 0 = G_{10}(\infty)\ . \tag{5.78d}$$

The above mentioned (5.76), (5.77) and (5.78), sets of equations are solved numerically in a manner similar to that of the transverse curvature problem of Sec. 5.6.1. The results are shown in Figs. 5.16 to 5.21.

The various boundary layer characteristics (5.8) for the effect of longitudinal curvature are

$$\begin{aligned} \tau_2\sqrt{(2\xi)}/(U_1^2r^j) & = F''(\xi, 0) \\ & = k_{ol}\,\tau_{2,0} + \xi(k_{1l}\,\tau_{2,1}+\beta_1k_{ot}\,\tau_{2,10}) + \cdots \end{aligned} \tag{5.79}$$

$$\begin{aligned} \frac{q_2\sqrt{(2\xi)}}{\sigma^{-1}(S_1-h_w)U_1r^j} & = G'(\xi, 0) \\ & = q_{2.0} + \xi(k_{1t}q_{2,1}+\beta_1k_{ot}q_{2,10}) + \cdots \end{aligned} \tag{5.80}$$

$$\delta_2^* \; U_1 r^j / \sqrt{(2\xi)} = -\int_0^\infty \eta + F' \; d\eta \tag{5.81}$$

$$= k_{01} \; \delta^*_{2,o} + \xi \, (k_{11} \; \delta^*_{2,1} + \beta_1 k_{01} \; \delta^*_{2,10}) + \cdots$$

$$\Delta_{\,2} U_1 r^j / \sqrt{(2\xi)} = \int_0^\infty (1-2f')F' - k_1 \, \eta \, f' \; d\eta \tag{5.82}$$

$$= K_{01} \, \Delta_{\,2,o} + \xi \, (k_{11} \, \Delta_{\,2,1} + \beta_1 \; k_{01} \Delta_{\,2,10}) + \cdots$$

Substituting (5.75) in (5.79) to (5.82) we get

$$
\begin{aligned}
\tau_{2,o} &= F_o''(0) \\
q_{2,o} &= G_o'(0) \\
\delta^*_{2,o} &= -\int_0^\infty \eta + F_o' \; d\eta \\
\Delta_{2,o} &= \int_0^\infty (1-2f_o')F_o' - \eta f_o' \; d\eta \, ,
\end{aligned}
\tag{5.83}
$$

$$
\begin{aligned}
\tau_{2,1} &= F_1'' \; (0) \\
q_{2,1} &= G_1' \; (0) \\
\delta^*_{2,1} &= -\int_0^\infty \eta + F_1' \; d\eta \\
\Delta_{2,1} &= \int_0^\infty (1-2f_o') \; F_1 - \eta \; f_o' \; d\eta
\end{aligned}
\tag{5.84}
$$

and

$$
\begin{aligned}
\tau_{2,1o} &= F_{1o}'' \; (0) \\
q_{2,1o} &= G_{1o}' \; (0) \\
\delta^*_{2,1o} &= -\int_0^\infty F_{1o}' \; d\eta \\
\Delta_{2,1o} &= \int_0^\infty (1-2f_o')F_{1o}' - 2f_1' F_o - \eta \; f_1' \; d\eta \, .
\end{aligned}
\tag{5.85}
$$

Here $\alpha = \lim_{\eta \to \infty} (\eta - f)$

$= \alpha_o + \xi \beta_1 \alpha_1 + \ldots$

with $\alpha_o = \lim_{\eta \to \infty} (\eta - f_o)$

$\alpha_1 = \lim_{\eta \to \infty} (-f_1)$ etc.

The above characteristics are also evaluated for various values of β_o and and are displayed in Figs. 5.22 to 5.24.

5.6.3 <u>External Vorticity</u>:

In this case we keep the non-homogenous terms in (5.2) which involve $B_1'(0)$ the problem for $u_2^{(v)}$ etc. reduce to

$$
\begin{aligned}
&(r^j u_2)_s + (r^j v_2)_N = 0 \\
&L_1(u_2) + L_2(u_1) - u_{2NN} = r^j V_2(s,o) \\
&L_1(h_o) + L_2(h_1) - \sigma^{-1} h_{2NN} = 0 \qquad (5.86) \\
&u_2(s,o) = 0 = v_2(s,o),\ u_2(s, N) = -r^j N,\ N \to \infty \\
&h_2(s, o) = h_2(s, \infty) = 0
\end{aligned}
$$

where $V_2(s, o) = \lim_{N \to \infty} (v_1 - N v_{1N})$.

The equation of continuity in (5.86) is satisfies by introducing the stream function

$$
\begin{aligned}
\psi_{2N} &= r^j u_2 \\
\psi_{2s} &= r^j v_2 .
\end{aligned}
\qquad (5.87)
$$

Further, we assume the following forms for ψ_2 and h_2

$$
\begin{aligned}
\psi_2(s, N) &= 2\xi\, F(\xi, \eta)/U_1^2 \\
h_2(s, N) &= G(\xi, \eta)\, (h_w - S_1)(2\xi)^{1/2}/U_1^2
\end{aligned}
\qquad (5.88)
$$

the velocity components are

$$u_2 = (2\xi)^{1/2} r^j F'/U_1$$
$$v_2 = -(2(1-\beta)F+2\xi F_\xi + \eta(\beta+j\Lambda_s-1)F')r^j/U_1 \qquad (5.89)$$

and the equations (5.86) become

$$F''' + fF'' - f'F' + 2(1-\beta)f''F + 2\xi(f_\xi F'' - f'F_\xi' - f_\xi' F' + f''F_\xi)$$
$$= \alpha(1+\Lambda_v) \qquad (5.90)$$

$$\sigma^{-1}G'' + fG' - (1+\Lambda-2\beta)f'G + 2\xi(f_\xi G' - f'G_\xi) \qquad (5.91)$$
$$= -Fg' + (1+\Lambda-2\beta)F'g - 2\xi(F_\xi g' - F'g_\xi)$$

$$2(1-\beta)F(\xi, o) + 2\xi F_\xi(\xi, o) = 0 \qquad (5.92)$$

$$F'(\xi, o) = 0, \quad F'(\xi, \eta) = -\tau, \quad \eta \to \infty$$
$$G(\xi, o) = 0 = G(\xi, \infty). \qquad (5.93)$$

where

$$\Lambda_v = (2\xi/\alpha)(d\alpha/d\xi)$$
$$= \frac{2\ d\alpha/ds}{\alpha U_1 r^j} \int_o^s U_1 r^{2j}\, ds \qquad (5.95)$$

will be called the principal function of vorticity interaction and arises from the second order changes in pressure induced upon the boundary layer by the interaction of the displacement thickness with vorticity.

Assuming the following expansions for F and G

$$F = F_o + \xi\beta_1 F_1 + \cdots\cdots$$
$$G = G_o + \xi\beta_1 G_1 + \cdots\cdots \qquad (5.96)$$

$$\frac{q_2 U_1}{(s_1 - h_w) r^j \sigma^{-1}} = G'(\xi, o)$$

$$= q_{2,o} + \xi \beta_1 \cdot q_{2,1} + \cdots \qquad (5.100)$$

$$\delta_2^* = -\int_o^\infty \eta + F' \, d\eta$$

$$= \delta_{2,o} + \xi \beta_1 \, \delta_{2,1} + \cdots \qquad (5.101)$$

$$\Delta_2 = \int_o^\infty (1-2f')F' - \eta f' \, d\eta$$

$$= \Delta_{2,o} + \xi \beta_1 \Delta_{2,1} + \cdots \qquad (5.102$$

Substituting the expansions (5.96) into above, the terms independent of ξ give

$$\begin{aligned} \tau_{2,o} &= F_o''(0) \\ q_{2,o} &= G_o'(0) \\ \delta_{2,o}^* &= -\int_o^\infty \eta + F_o' \, d\eta \\ \Delta_{2,o} &= \int_o^\infty (1-2f_o')F_o' - \eta f_o' \, d\eta \end{aligned} \qquad (5.103)$$

and the terms of order ξ are

$$\begin{aligned} \tau_{2,1} &= F_1''(0) \\ q_{2,1} &= G_1'(0) \\ \delta_{2,1}^* &= -\int_o^\infty F_1' \, d\eta \\ \Delta_{2,1} &= \int_o^\infty (1-2f_o')F_1' - 2f_1'F_o - \eta f_1' \, d\eta . \end{aligned} \qquad (5.104)$$

The above boundary layer characteristics are studied in the same way as the previous two problems and the results are shown in Figs. 5.29 to 5.32.

5.6.4 Stagnation Enthalpy Gradients:

Keeping the nonhomogeneous terms in Eq. (5.2) which are proportional to $S_1'(0)$ we have

$$
\left.
\begin{aligned}
&(r^j u_1)_s + (r^j v_1)_N = 0 \\
&L_1(u_2) + L_2(u_1) - u_{2NN} = 0 \\
&L_1(h_2) + L_2(h_1) - \sigma^{-1} h_{2NN} = 0
\end{aligned}
\right\} \quad (5.105)
$$

$$u_2(s,o) = v_2(s,\ o) = u_2(s,\ \infty) = 0$$

$$h_2(s,\ o) = 0, \qquad h_2(s,\ N) = \Psi_1(s,\ N), \quad N \to \infty .$$

Now it is to be noted that the momentum problem is a homogenous one with homogenous boundary conditions and its solution is trivial

$$u_2(s,\ N) = 0, \quad v_2(s,\ N) = 0 \qquad (5.106)$$

and the energy equation reduces to

$$L_1(h_2) - \sigma^{-1} h_{2NN} = 0. \qquad (5.107)$$

Assuming the following form for h_2

$$h_2(s,\ N) = (2\xi)^{1/2}\, G(\xi, \eta) \qquad (5.108)$$

the energy equation (5.107) reduces to

$$\sigma^{-1} G'' + fG' + (2\Lambda - 1)f'G + 2\xi(f_\xi G' - f'G_\xi) = 0 \qquad (5.109)$$

$$G(0) = 0, \quad G(\eta) = \eta - \alpha, \quad \eta \to \infty.$$

To study (5.109) we assume the following expansion for G

$$G(\xi, \eta) = G_o(\eta) + \xi \beta_1 G_1(\eta) + \cdots \qquad (5.110)$$

Substituting (5.110) in (5.109) and collecting the coefficients of various powers of ξ, the terms independent of ξ give

$$Z_{1/2}\,(G_o) = 0 \tag{5.111}$$

$$G_o \qquad G_o(0) = 0,\; G_o(\infty) = \eta - \alpha_o$$

and the terms of order ξ give

$$Z_{3/2}\,(G_1) = 0 \tag{5.112}$$
$$G_1(0) = 0 \qquad G_1(\infty) = -\,\alpha_1.$$

These two sets of equations (5.111) and (5.112) are integrated numerically and results are shown in Fig. 5.33 to Fig. 5.34.

To study the various boundary layer characteristics (5.8) we note that $\tau_2 = \delta_2^* = \Delta_2 = 0$ and

$$q_2/(-\sigma^{-1} r^j U_1) = G(\xi,\, o) \tag{5.113}$$
$$= q_{2,o} + \xi\beta_1\, q_{2,1} + \ldots$$

The expressions for q's are

$$q_{2,o} = G_o'\,(0)$$
$$q_{2,1} = G_1'\,(0) \tag{5.114}$$

The results for (5.114) are shown in Fig. 5.35.

5.6.5 Displacement Speed Problem:

The remaining non-homogenous terms give the problem for $u_2^{(d)}$ etc:

$$(r^j u_2)_s + (r^j v_2)_N = 0$$
$$L_1(u_2) + L_2(u_1) - u_{2NN} = (U_1 U_2)_s \tag{5.115}$$
$$L_1(h_2) + L_2(h_1) - \sigma^{-1} h_{2NN} = 0$$
$$u_2(s,\, o) = 0 = v_2(s,\, o),\; u_2(s, u_2(s,\, \infty) = U_2$$
$$h_2(s,\, o) = 0 = h_2(s,\, \infty).$$

In this case the second order stream function is given by

$$\psi_{2N} = r^j u_2$$
$$\psi_{2s} = r^j v_2 \qquad (5.116)$$

The functions ψ_2 and h_2 are assumed to be of the form

$$\psi_2(s, N) = \sqrt{2\xi}\, F(\xi, \eta)$$
$$h_2(s, N) = (h_w - S_1) G(\xi, \eta) \qquad (5.117)$$

and this gives the velocity components

$$u_2 = U_1 F' \qquad (5.118)$$

$$v_2 = -U_1 r^j \left(F + 2\xi F_\xi + \eta(\beta + j\Lambda_s + 1)F'\right)/\sqrt{(2\xi)}.$$

Now the equations (5.115) with the help of (5.117) and (5.118) reduce to

$$F''' + F'' - 2\beta f' F' + f'' F + 2\xi (f_\xi F'' - f' F'_\xi - f'_\xi F' + f'' F_\xi)$$
$$= \lambda(2\beta + \Lambda_d) \qquad (5.119)$$

$$\sigma^{-1} G'' + fG' - \Lambda f' G + 2\xi (f_\xi G' - f' G_\xi) = -Fg'$$
$$+ \Lambda F' g + 2\xi (F' g_\xi - F_\xi g'),$$

$$F(\xi, o) + 2\xi F_\xi(\xi, o) = 0$$
$$F'(\xi, o) = 0; \quad F'(\xi, \infty) = \lambda(\xi)$$
$$G(\xi, o) = 0 = G(\xi, \infty).$$

Here the quantity

$$\lambda(\xi) = U_2/U_1 \qquad (5.120)$$

is the ratio of displacement speed to velocity of slip of first order outer flow and

$$\Lambda_d = (2\xi/\lambda)(d\lambda/d\xi)$$

$$= \frac{2d\lambda/dx}{U_1 r^{2j}} \int_0^s U_1\ r^{2j}\ ds \qquad (5.121)$$

will be called the principal displacement speed function.

Now to study the displacement problem we need to know $U_2(s, o)$, which may be determined by various methods mentioned in the Chapter 1. Suppose that this has been done, so that the change in surface speed is

$$U_2(s, o) = x^m \sum_{n=0} \lambda_{2n}\ x^n . \qquad (5.121)$$

This expression (5.121) includes the very important cases of: the flat plate (m = 0) for which it vanishes ($\lambda_{2n} = 0$), the plane or axisymmetric flow very near to the point of stagnation. In the stagnation region (m = 1) of parabolic cylinder Van Dyke (1964) found that $\lambda_{2o} = -0.61$.

Now (5.121) and (5.19) gives

$$\lambda(\xi) = \lambda_o + \lambda_1\xi + \cdots \qquad (5.122)$$

and assuming the following forms for F and G

$$\begin{aligned} F &= \lambda_o F_o + \xi(\lambda_1 F_1 + \lambda_o\beta_1 F_{10}) + \cdots \\ G &= \lambda_o G_o + \xi(\lambda_1 G_1 + \lambda_o\beta_1 G_{10}) + \cdots \end{aligned} \qquad (5.123)$$

the equations (5.119) gives

$$\begin{aligned} Y_o(F_o) &= -2\beta_o \\ Z_o(G_o) &= -F_o g_o' \\ F_o(0) = 0 &= F_o'(0)\ , \quad F_o'(\infty) = 1 \\ G_o(0) = 0 &= G_o(\infty), \end{aligned} \qquad (5.124)$$

$$
\begin{aligned}
Y_1(F_1) &= -2(1+\beta_o) \\
Z_1(G_1) &= -3F_1 g_o' \\
F_1(0) &= 0 = F_o'(0); \quad F_1'(\infty) = 1 \\
G_1(0) &= 0 = G_1(\infty)
\end{aligned}
\tag{5.125}
$$

and

$$
\begin{aligned}
Y_1(F_{10}) &= -2-3f_1F_o'' + 2(1+\beta_o)f_1'F_o'-f_1''F_o+2f_o'F_o' \\
Z_1(G_{10}) &= -3f_1G_o' - 3F_{10}g_o' + 2F_o'g_1 - F_og_1' \\
F_{10}(0) &= F_{10}'(0) = F_{10}'(\infty) = 0 \\
G_{10}(0) &= 0 = G_{10}(\infty).
\end{aligned}
\tag{5.126}
$$

It is interesting to note that the equations (5.124), (5.125) and (5.126) can be solved analytically in closed form in terms of first order boundary layer solutions. The solution for equations (5.124) is

$$F_o = (f_o + \eta f_o')/2 \tag{5.127a}$$

$$G_o = \eta g_o'/2 \tag{5.127b}$$

for (5.125) is

$$F_1 = (4-\beta_o)f_1/2 + (f_o+3\eta f_o')/4 \tag{5.128a}$$

$$G_1 = (4-\beta_o)g_1/2 + 3\eta g_o'/4 \tag{5.128b}$$

and for (5.126) is

$$F_{10} = (f_1 + \eta f_1)/2 \tag{5.129a}$$

$$G_{10} = \eta g_1'/2\ . \tag{5.129b}$$

The solutions (5.127), (5.128) and (5.129) are evaluated from

corresponding first order solutions and are displayed in Fig. 5.36 to Fig. 5.41. The final form of the solution to F and G is givehby by

$$
\begin{aligned}
F &= \lambda_o(f_o + \eta f_o')/2 + \xi\left[\lambda_1(f_o+3\eta f_o')/2\right. \\
&\quad \left. + \lambda_1(4-\beta_o)f_1 + \lambda_o\beta_1(f_1+\eta f_1')\right]/2 + \cdots \\
G &= \lambda_o\eta g_o'/2 + \left[\lambda_1\eta g_o'/2 + \lambda_1(4-\beta_o)g_1\right. \\
&\quad \left. + \lambda_o\beta_1\eta g_1'\right]/2 + \cdots
\end{aligned}
\tag{5.130}
$$

and may be calculated in specific cases from the first order solution. In this case various boundary layer characteristics are

$$\tau_2\sqrt{2\xi}/(U_1^2 r^j) = F''(\xi, 0) \tag{5.131}$$

$$= \lambda_o\tau_{2,o} + \xi(\lambda_1\tau_{2,1} + \beta_1\lambda_o\tau_{2,10}) + \cdots$$

$$\frac{q_2\sqrt{(2\xi)}}{\sigma^{-1}(S_1-h_w)U_1 r^j} = G'(\xi, 0) \tag{5.132}$$

$$= \lambda_o q_{2,o} + \xi(\lambda_1 q_{2,1} + \beta_1\lambda_o q_{2,10}) + \cdots$$

$$\delta_2^* U_1 r^j/\sqrt{(2\xi)} = \int_0^\infty \lambda - F' \, d\eta \tag{5.133}$$

$$= \lambda_o\delta_{2,o}^* + \xi(\lambda_1\delta_{2,1}^* + \beta_1\lambda_o\delta_{2,10}^*) + \cdots$$

$$\Delta_2 U_1 r^j/\sqrt{(2\xi)} = \int_0^\infty (1-2f')F' + \lambda f' \, d\eta \tag{5.144}$$

$$= \lambda_o\Delta_{2,o} + \xi(\lambda_1\Delta_{2,1} + \beta_1\lambda_o\Delta_{2,10}) + \cdots.$$

Substituting the expansions (5.122) and (5.123) in above and collecting various universal functions we get

$$\tau_{2,o} = F_o''(0)$$

$$q_{2,o} = G_o'(0)$$

$$\delta^*_{2,o} = \int_0^\infty 1 - F_o' \, d\eta \qquad (5.145)$$

$$\Delta_{2,o} = \int_0^\infty (1-2f_o') \; F_o' + f_o' \;\; d\eta \; ,$$

$$\tau_{2,1} = F_1''(0)$$

$$q_{2,1} = G_1'(0)$$

$$\delta^*_{2,1} = \int_0^\infty 1-F_1' \;\; d\eta \qquad (5.146)$$

$$\Delta_{2,1} = \int_0^\infty (1-2f_o') \; F_1' + f_o' \; d\eta$$

and

$$\tau_{2,10} = F_{10}'' (0)$$

$$q_{2,10} = G_{10}' (0)$$

$$\delta^*_{2,10} = \int_0^\infty - F_{10}' \, d\eta$$

$$\Delta_{2,10} = \int_0^\infty (1-2f_o') \; F_{10} - 2f_1' . F_o - f_1' \; d\eta .$$

These characteristics (5.145), (5.156) and (5.147) are also evaluated from the first order boundary layer solution, the results of calculation are displayed in Figs. 5.42 to 5.44.

5.7 Discussion

The higher order terms in the asymptotic expansion for large Reynolds number represents the corrections to the leading (first order boundary layer) approximation. The utility of these corrections depend upon the behavior of the first order boundary layer itself. In first order boundary layer, the function f_0 satisfies the Falkner-Skan equation whose solution has been studied extensively in the literature. Solutions to the Falkner-Skan equation with exponential decay at infinity exists (see, Chapter 4) for $-0.198838 \leq \beta_0 < \infty$ with $f_0''(0) \geq 0$. However, for most of the problems of interest $\beta_0 \leq 2$. The velocity profile f_0' vs η for various values of β_0 is shown in Fig. 5.1. The Fig. 5.5 displays the skin friction $\tau_{1,0} \equiv f_0''(0)$ vs β_0. The quantity $f_0''(0)$ decreases as β_0 decreases and becomes zero at $\beta_0 = -0.198838$ (this later point will be referred later as the classical separation point and cited as CPP). The corresponding energy equation for g_0 is also studied extensively (see, Pai 1956 and Schlichting 1968). Some of the solutions for temperature profile g_0 vs η obtained in the present work are shown in Fig. 5.2. Heat transfer rates are shown in Fig. 5.6. The magnitude of heat transfer $q_{1,0}$ decreases as β or ζ decreases.

The solutions for f_1' and g_1 (the second term of Gortler power series in the first order boundary layer theory) are shown in Figs. 5.4 and 5.5 respectively. Near CPP the functions f_1 and g_1 becomes very large compared to f_0 and g_0. Further, the function

$f_1''(0)$ and $g_1'(0)$ are shown in Figs. 5.5 and 5.6, indicate that these increases as β_0 decreases. The Fig. 5.5 also shows the displacement and momentum thicknesses due to similar and non-similar terms. For few cases these equations are studied by Gortler and Sparrow, the comparison of our results with their calculations is as follows

			$\beta_0 = 0$	$\beta_0 = 1$
Gortler (1957)	$f_1''(0)$	=	1.032361	0.493840
Present work		=	1.0323626	0.4938403
			$\beta_0 = 1$ and $\sigma = 1$	
Sparrow (1958)	$g_1'(0)$	=	- 0.062118	
Present work		=	- 0.0621187.	

Furthermore, the convergence of the series for f is good for positive values of β_0 as also shown by Gortler) but for negative values of β_0 and particularly near CPP is poor. The similar comments also apply to series for g. At first sight one may expect that the expansions for f and g are not uniformly valid near CPP. It is shown later that near CPP (i) the convergence of second order Gortler Power power series is also poor (ii) even the convergence of leading terms (in the first and second order theory for $\quad = 0$) is very poor in some of the second order effects. Hence there seems a more basic nonuniformity near CPP in inner asymptotic expansion itself. The reason is this that our inner expansion is not valid near CPP or ? any seperated flow - as the appropriate Euler limit of Navier-Stokes equations is not known (see Chapter 2). Thus we see that the boundary layer expansion is not uniformly valid in ~~in~~ γ flow involving separation.

Now we proceed to study the various second order corrections to the first order boundary layer theory. The solution to transverse curvature problem governed by the sets of equations (5.76), (5.77) and (5.78) are shown in Fig. 5.7 to 5.12. The Fig. 5.7 for the solution of the first term F_o in the second order Gortler power series show that variation in velocity profile for various values of β_o are small while in higher order terms F_1' and F_{1o}' (see, Figs. 5.9 and 5.11) the variations are large, particularly near CPP. Similar remarks applies to the corresponding temperature profiles G_o, G_1, and G_{1o} shown in Figs. 5.8, 5.10 and 5.12 respectively. The skin friction for the three sets of equations is shown in Fig. 5.13. As β_o decreases the quantities $\tau_{2,o}$ and $\tau_{2,1}$ increases where $\tau_{2,1o}$ first increases and then decreases to take large negative values near CPP. The Fig. 5.14 shows the heat transfer for the three sets. The function $-q_{2,o}$, $-q_{2,1}$ and $q_{2,1o}$ increases as β_o decreases. The effect of transverse curvature on displacement and momentum thicknesses is shown in Fig. 5.15. Finally for a given pressure gradient β_o, the transverse curvature increases the skin friction and heat transfer.

The problem of longitudinal curvature has been controvertial (see, Chapter 1). Let us first study the leading term described by equations (5.76). The velocity and temperature profile are shown in Figs. 3.15 and 3.16. The momentum equation (5.76a) has also been studied by Narasimha and Ojha (1967). Our

results are in agreement with these authors except those of momentum thickness (see Fig. 5.23). The comparison of our results with various authors is as follows

Source	$\tau_{2,0}$
1. $\beta_o=0$	
Murphy (1953)	1.753
Tani (1964)	1.449
Yen and Toba (1961)	- 0.975
Hayasi (1963)	1.449
Murphy (1965)	0.997
Schultz-Grunow and Breuer (1965)	1.442
Narasimha and Ojha (1967)	1.4469
Present work	1.4469667
2. $\beta_o = 1$	
Van Dyke (1962b)	1.913255
Narasimha and Ojha (1967)	1.9132
Present work	1.9132549.

The comparison of heat transfer results for $\sigma = 0.7$ is as follows:

Source	$q_{2,0}$
1. $\beta_o = 0$	
Schultz-Grunow and Breuer (1965)	0.245
Present work	0.2508955
2. $\beta_o = 1$	
Van Dyke (1962b)	0.12811
Present work	0.1281108

It is to be noted that the velocity and temperature profiles shown in Figs. 5.15 and 5.16 become very large near CPP. The velocity and temperature profiles for nonsimilar sets of equations (5.77) and (5.78) are shown in Figs. 5.17 to 5.20 and these also becomes large near CPP. The skin friction for the three sets (5.76) to (5.78) is shown in Fig. 5.22. The fig 5.22 shows that $\tau_{2,0}$, $\tau_{2,1}$ and $\tau_{2,0}$ increases as β_0 decreases. The similar trend is observed for heat transfer results displayed in Fig. 5.23, when Prandtl number is prescribed. If β_0 is held fixed the heat transfer increases as Prandtl number increases. The Fig. (5.24) shows the effects of longitudinal curvature on displacement and momentum thicknesses. Lastly, regarding the effect of longitudinal curvature, it is clear that convex curvature reduces both the skin friction and the heat transfer.

The subject the effect of vorticity in the basic flow upon the classical boundary layer has been the subject of prolonged controversy. (see, Chapter 1). Let us first study our first term in Gortler power series expansion for the vorticity effects described by equations (5.97). The velocity and temperature profiles for these equations are shown in Figs. 5.25 and 5.26. The skin friction and heat transfer are shown in Figs. 5.29 and 5.30. The comparison with various authors is an follows:

Skin friction:

Source	$-F_0''(0)$
1. $\beta_0 = 0$	
Van Dyke (1962b)	3.1260

*Glauert (1957)	0.795
Murray (1961)	3.1259
Present work	3.125983
2. $\beta = 0.5$	
*Rott and Lenard (1959)	0.649
*Kemp (1959)	0.650
Van Dyke (1962b)	1.76861
Present work	1.768597
3. $\beta_o = 1.0$	
*Stuart (1959)	0.6078
Van Dyke (1962b)	1.40652
Present work	1.406543

Heat transfer:

Source	$G_o'(0)$	
	$\beta_o = 0$	$\beta_o = 0.5$
1. $\sigma = 0.7$		
Van Dyke (1962b)	0.91117	0.38270
Present work	0.911168	0.38269
2. $\sigma = 1.0$		
*Ovchinnikov (1960)	0.340	
Present work	1.042645.	

The authors marked with '*' above have neglected the induced pressure gradient by the interaction of external vorticity with classical boundary layer. So in some sense they have solved the momentum equation (5.97a) with α_o omitted. Thus if a particular integral is added to the results of above authors the present results can be obtained. The Fig. 5.30 displays the heat transfer $q_{2,o}$ for

various values of β_o. As β_o decreases $q_{2,o}$ increases. Further, the velocity and temperature profiles for the second term in Gortler power series described by (5.98) are shown in Fig.5.27 and 5.28. The skin friction and heat transfer results shown in Figs. 5.29 and 5.30. The effect of vorticity on displacement and momentum thicknesses is shown in Fig. (5.31). Finally the effect of vorticity is to decrease skin friction for all values of β_o. Further, the external vorticity also reduces the heat transfer except for $\beta_o > 1$ where it increases. Also near CPP the velocity and temperature becomes large.

The effect of stagnation enthalpy on the boundary layer is described by equations (5.111) and (5.112). The Figs. 5.32 and 5.33 shows the temperature profiles for the above equations. The heat transfer is shown in Fig. 5.34, which shows that as β_o decreases the function $q_{2,o}$ decreases while $q_{2,1}$ increases. The effect of stagnation enthalpy gradients in basic flow is to reduce the heat transfer. However near CPP the temperature becomes very large.

Lastly for the displacement problem (Sec. 5.6.5) the exact close form solutions are given in terms of the first order boundary layer. The solutions (5.127), (5.128), (5.129) for the sets of equations, (5.124), (5.125) and (5.126) are shown in Figs. 5.36 to 5.41. The velocity and temperatures again increases near CPP. The skin friction is shown in Fig. 5.42. The figure shows $\tau_{2,o}$ decreases as β_o decreases while $\tau_{2,1}$ and $\tau_{2,1o}$ increases.

Similar comments apply to the heat transfer results shown in Fig. 5.43. The Fig. 5.44 shows the effect of displacement speed on the displacement and momentum thicknesses. The effect of displacement speed is to increase the skin friction and heat transfer.

5.8 Conclusions:

The present solution for the second order effects show that the transverse curvature increases the skin friction and heat transfer. The convex longitudinal surface curvature decreases the skin friction and heat transfer. The effect of external vorticity is to decrease the skin friction. The heat transfer is also decreased except for $\beta_0 > 1$ where it increases. The affect of stagnation ~~on~~ enthalpy is to decrease heat transfer. Finally, the displacement speed increases the skin friction and heat transfer.

Lastly, the results indicate that the convergence of Gortler power series for the first and the second order boundary layer functions becomes poor as favourable pressure gradient diminshes. Furthermore, in adverse pressure gradient (particularly near $\beta_0 = -0.198838$) some of the second order boundary layer quantities become~~s~~ very large compared with the corresponding first order boundary boundary layer quantities. This indicates that the boundary layer (inner) expansion is not uniformly valid near the separation.

REFERENCES

1 Adams, E.W.
1963 Heat transfer in laminar flows of incompressible fluid with $Pr \to 0$ and $Pr \to \infty$. NASA TND 1257.

2 Afzal, N.
1967 Some studies in magneto-fluid-dynamic boundary layers. M.E. Project Report, Department of Aeronautical Engineering, Indian Institute of Science, Bangalore, India.

3 Babister, A.W.
1967 Transcendental Functions Satisfying Nonhomogenous Linear Differential Equations. The Macmillan Co.,New York.

4 Bush, W.B.
1964 Local similarity concept of the boundary layer equations. AIAA Journal, 2, 1857.

5 Chapman, S. and Cowling, T.G.
1939 Mathematical Theory of Non-Uniform Gases. Cambridge University Press.

6 Cheng, H.K.
1963 The blunt body problem in hypersonic flow at low Reynolds number. Cornell Aeronautical Lab. Rep. AF-1285-A-10.

7 Cheng, H.K.
1966 Viscous hypersonic blunt body problem and Newtonian theory. In Proceedings of International Symposium on Fundamental Phenomena in Hypersonic Flow p. 140. Cornell University Press.

8. Chow, R.R. and Ting, L.
1961 Higher Order theory of curved shock J. Aero. Sci. 28, 428-430.

9 Cole, J.D.
1968 Perturbation Methods in Applied Mathematics.Blaisdell Publishing Company, Massachusetts.

10 Coppel, W.
1960 On a differential equation of boundary layer theory. Phil. Trans. 253, 101-136.

11. Crocco, L.
1932 Sulla transmissione der calore da una lamina piana a un fluido scorreute and alta velocita. L'Aerotechnica 12, 181-197.

12 Davis, R.T. and Flugge-Lotz, I.
1964a The laminar compressible boundary layer in the stagnation point region of an axisymmetric blunt body including the second order vorticity interaction. Int. J. Heat Mass Transfer 7,341-370.

13 Davis, R.T and Flugge-Lotz, I.
1964b Second order boundary layer effects in hypersonic flow part asixymmetric blunt bodies J. Fluid Mech. 20, 593-623.

14. Devan, L. and Oberai, M.M.
1964 Approximate solution of second order boundary layer equations. AIAA Journal, 2, 1838-1840.

15 Devan, L.
1965 Approximate solution of second order boundary layer equations. AIAA Journal 3, 2197-2202.

16. Edwards, D.K. and Tellep, D.M.
1961 Heat transfer in low Prandtl number flows with variable thermal properties. ARS Journal 31, 652-654.

17 Ellinwood, J.B. and Mirels, H.
1969 Axisymmetric hypersonic flow with mass transfer and large transverse curvature AIAA Journal (to appear).

18 Erdelyi, A , Magnus, W., Oberhettinger, F. and Tricomi, F.G.
1954 Higher Transcendental Functions. Bateman Manuscript Project, 3 Vol. 1 McGraw Hill, New York.

19 Fvans, H.L.
1967 Laminar Boundary Layer Theory. Addison-Wesley Publishing Company, London.

20 Falkner, V.M. and Skan, S.W.
1931 Some approximate solutions of the boundary layer equations. Phil. Mag. 12, 865-896; ARC RM 1314 (1930).

21 Fannelop, T.K. and Flugge-Lotz, I.
1965 Two dimensional hypersonic stagnation flow at low Reynolds number. Z. Flugwiss. 13, 282-296.

22 Fannelop T.K. and Flugge-Lotz, I.
1966 Viscous hypersonic flow over simple blunt bodies; comparison of a second order theory with experiments. J. De. Mecanique 5, 69-100.

23 Ferri, A. and Libby, P.A.
1954 Note on an interaction between the boundary layer and the inviscid flow. J. Aero. Sci. 21, 130.

24 Ferri, A., Zakkay, V. and Ting, L.
1962 On blunt body heat transfer at hypersonic speeds and low Reynolds number. J. Aero/Space Sci. 29, 882.

25 Kao, H.C.
1964 Hypersonic viscous flow near the stagnation stream line of a blunt body, II Third order boundary layer theory and comparison with other methods. AIAA Journal 2, 1898-1906.

26 Germani, P. and Guiraud, J.P.
1962 Conditions de choc et structure des ondes dc choc dans un ecoulement stationnaire de fluide dissipatif. ONERA Publication No. 105.

27 Glaucrt, M.B.
1957 The boundary layer in simple shear flow past a flat plate 24, 848-849.

28 Glauert, M.B. and Lighthill, M.J.
1955 The axisymmetric boundary layer on a long thin cylinder. Proc.Roy.Soc.A230, 188-203.

29. Goddard, J.D. and Acrivos, A.
1966 Asymptotic expansions for laminar forced convection heat and mass transfer, part 2. Boundary layer flows. J. Fluid Mech. 24, 339-366.

30 Gortler, H.
1957 A new series for the calculation of steady laminar boundary layer flows J. Math. Mech. 6, 1-66.

31 Gortler, H.
1957 Zahleutafeln universeller Funktionen zur neuen Reihe fur die Berechung Laminarer Grenzschichten. Bericht No.34 of Deutsche Versuchsanstalt fur Luftfahrt.

32 Hardy, G.H.
1949 Divergent Series, p. 178-181. Oxford Clarendon Press.

33 Hayasi, N.
1963 Effects of surface curvature on laminar boundary layer flow. AIAA Journal 1, 914-915.

34 Hayes, W.D.
1956 On laminar boundary layer with heat transfer. Jet Propulsion 26, 270-274.

35 Hayes, W.D. and Probstein, R.F.
1959 Hypersonic Flow Theory. Academic Press, New York.

36 Honda, M. and Kiyokawa, Y.
1969 An analysis of second order effect on laminar boundary layer flow. J. Fluid Mech. 35-145-169.

37 Jaffe, N.A. and Okamura, T.T.
1968 The transverse curvature effect on the incompressible laminar boundary layer for longitudinal flow over a cylinder. ZAMP 19, 564-574.

38 Kaplun, S.
1954 The role of coordinate system in the boundary layer theory. ZAMP. 5, 111-135.

37 Kaplun, S.
1957 Low Reynolds number flow past a circular cylinder. J.Math.Mech.6, 593-603.

40 Kaplun, S.
1967 Singular Perturbation and Fluid Mechanics. Eds: Lagerstrom, P A., Howard, L.N. and Liu, C.S. Academic Press, New York.

41 Kaplun, S. and Lagerstrom. P.A.
1957 Asymptotic expansions of Navier-Stokes solutions for small Reynolds number. J.Math. Mech 6, 585-593.

42 Kemp, N.H.
1959 Vorticity interaction at an axisymmetric stagnation point in a viscous incompressible fluid. J.Aero/Space, Sc.26, 543-544.

43 Kuerti, G.
1950 Laminar boundary layer in compressible flow. In Advances in Applied Mechanics Vol II, edited by R.Von Mises and T. Von Karman, p. 23. 91. Academic Press, New York.

44 Lagerstrom, P.A.
1964 Laminar flow theory. In Theory of Laminar Flows (High Speed Aerodynamics and Jet Propulsion Vol. 4) edited by F.K. Moore. Princeton University Press.

45 Lagerstrom, P.A. and Cole, J.D.
1955 Examples illustrating expansion procedures for Navier-Stokes equations. J. Ral. Mech. Anal. 4,817-882.

46 Lagerstrom, P.A., Howard, L.N. and Liu, C.S.
1967 Singular Perturbation and Fluid Mechanics. Collected work of Saul Kaplun. Academic Press, New York.

47 Lenard, M.
1961 Stagnation Point of a Variable Property Fluid at Low Reynolds numbers. Ph.D. Thesis, Cornell University.

48 Lees, L.
1956 Laminar heat transfer over blunt nosed bodies at hypersonic flight speeds. Jet Propulsion 26, 259.

49 Li, T.Y.
1955 Simple shear flow past a flat plate in an incompressible fluid of small viscosity. J. Aero. Sc.22, 651-652.

50 Li, T.Y.
1956 Effects of free stream vorticity on the behaviour of a viscous boundary layer. J. Aero. Sc. 23, 1128-1129.

51. Li, T. and Gross, J.
1964 Transverse curvature effects in axisymmetric hypersonic boundary layers. AIAA Journal, 2, 1868-1869.

52 Liepmann, H.W., Narasimha, R. and Chahine, M.T.
1962 The structure of a plane shock layer. Phy. Fluids 5, 1313.

53 Lighthill, M.J.
1950 Contribution to the theory of heat transfer through a laminar boundary layer. Proc. Roy. Soc. A202, 359.

54 Magnus, W., Oberhettinger, F. and Soni, R.P.
1908 Formulas and Theorems for the Special Functions of Mathematical Physics. B. Eckmann und B.L. Van der Waerden.

55 Malsen, S.H.
1963 Second order effects in laminar boundary layers. AIAA Journal 1, 33.

56 Massey, B.S. and Clayton, B.R.
1965 Laminar boundary layers and their separation from curved surfaces. J. Basic Engg. ASME Trans. 87D, 483-494.

57 Meksyn, D.
1961 New Methods in Laminar Boundary Layer Theory. Pergamon Press, Oxford.

58 Merk, H.J.
1959 Rapid calculations for boundary layer transfer using wedge solutions and asymptotic expansions. J. Fluid Mech. 5, 460-480.

59 Mitchell Thomass Jr.
1962 Transport properties of high temperature gases. In Proceedings of Fourth Biennial Gas Dynamics Symposium, edited by A.B. Cambel, T.P. Anderson and M.M. Slawsky, p. 89-108. Northwestern University Press.

60 Morgan, C.W., Pipkin, A.C. and Warner, W.H.
1958 On heat transfer in laminar boundary layer flow of liquids having very small Prandtl number. J. Aero. Sc.25, 173-180.

61 Moore, F.K.
1964 Hypersonic flow theory. In Laminar Flow Theory (High Speed and Jet Propulsion Vol. 4) edited by F.K. Moore. Princeton University Press.

62 Murphy, J.S.
1953 Some effects of surface curvature on laminar boundary layer flow. J. Aero. Sc. 20, 338-344.

63 Murphy, J.S.
1962 Comments on a theory of two dimensional laminar boundary layer over a curved surface. J. Aero. Sc. 29, 366-367.

64 Murphy, J.S.
1965 Extensions of the Falkner-Skan similar solutions to flows with surface curvature. AIAA Journal 3, 2043-2049.

65 Murray, J.D.
1961 The boundary layer on a flat plate in a stream with shear flow. J. Fluid Mech. 11, 309-316.

66 Narasimha, R.
1964 On the incomplete gamma function with one negative argument. Report AE 123 A.Dept. of Aeronautical Engg., I.I.Sc. Bangalore, India.

67 Narasimha, R. and Afzal, N.
1969 Laminar boundary layer on a flat plate at low Prandtl number. Report AE 171A, Dept. of Aeronautical Engineering, I.I.Sc. Bangalore.

68 Narasimha, R. and Ojha, S.K.
1967 Effect of longitudinal surface curvature on boundary layer. J. Fluid Mech. 29, 187-199.

69 Narasimha, R. and Vasantha, S.S.
1966 Laminar boundary layer on a flat plate at high Prandtl number, ZAMP 17, 582-592.

70 Oberai, M.M.
1963 Ecoulement Hypersonique Visqueux Autour d'un Obstacle de Revolution, a nez Arrondi, en Atmosphere Moderment Rarefiee. Ph.D. Thesis Universite de Paris.

71 Oberai, M.M.
1964 Ecoulement hypersonique visqueux autour d'un obstacle de revolution, a nez arrondi, en atmosphere moderment rarefiee. J. de. Mechanique 3, 143-214.

72 Ovchinnikov, O.N.
1960 The laminar boundary layer on a flat plate in a inhomogenous flow. Soviet Physics - Technical Physics 5, 590.

73 Pai, S.I.
1956 Viscous Flow Theory I-Laminar flow. D.Van Nostrand Company Inc. London.

74 Peng, T. and Pindroh, A.L.
1962 An improved calculation of gas properties at high temperatures. In. Proc. Fourth Bienniel Gas Dynamics Symposium edited by A.B. Cambel et.al. Northwestern University Press, Evanston.

75 Pohlhausen, E.
1921 Der Warmeaustausch zwischen festen Korpern und Flussigkeiten mit Kleiner Reibung und Kleiner Warmeleitung - ZAMM 1, 115-120.

76 Prandtl, L.
1928 Motion of fluids with very little viscosity. Tech. Memo. NACA 452.

77 Prandtl, L.
1935 The mechanics of viscous fluids. In:Aerodynamic Theory, Division G edited by W.F.Durand, Vol 3, p.34-208. Springer, Berlin.

78 Probstein, R. and Elliot, D.
1956 The transverse curvature effect in compressible axially symmetric laminar boundary layer flow. J. Aero. Sci 23, 208-224, 236.

79 Rott, N. and Lenard, M.
1959 Vorticity effect on stagnation point flow of a viscous incompressible fluid. J. Aero/Space Sci. 26,452.

80 Schlichting, H.
1962 Boundary Layèr Theory. Fourth Edition p. 159. McGraw Hill Book Company, New York.

81 Schlichting, H.
1968 Boundary Layer Theory (Sixth Edition). McGraw Hill Book Company. New York.

82 Schultz-Grunow, F. and Breuer, W.
1965 Laminar boundary layer on cambered walls. In: Basic Developments in Fluid Dynamics (ed: M.Holt). Academic Press, New York.

83 Seban, R.A. and Bond, R.
1951 Skin friction and heat transfer characteristics of a laminar boundary layer on a cylinder in axial incompressible flow. J.Aero.Sc. 18, 671-675.

84 Smith, A.M.O.
1954 Improved solutions of the Falkner-Skan Boundary Layer equations. IAS Fairchild Fund Preprint FF-10.

85 Sneddon, I.N.
1961 Special Functions of Mathematical Physics and Chemistry. Interscience Publishers Inc. New York.

86 Sparrow, E.M.
1955 Analysis of laminar forced convection heat transfer in entrance region of a flat rectangular duct. NACA TN 3331.

87 Sparrow, E.M.
1958 The thermal boundary layer on a non-isothermal surface with nonuniform free stream. J. Fluid Mech. 4, 321-329.

88 Sparrow, E.M. and Gregg, J.L.
1958 Viscous dissipation in low Prandtl number boundary layer flow, J.Aero. Sci. 25, 717-718.

89 Sparrow, E.M. and Gregg, J.L.
1959 Details of exact low Prandtl number boundary layer solutions for forced and for free convection, NASA Memo 2-27-59E.6

90 Spence, D.A.
1960 A note on the recovery and Reynolds analogy factors in laminar flat plate flow. J. Aero. Sci. 27, 878-879.

91 Stewartson, K.
1955 The asymptotic boundary layer on a circular cylinder in axial incompressible flow. Quart. J. App. Math. 13, 113-122.

92. Stewartson, K.
1964 Viscous hypersonic flow past a slender cone. Phy. Fluids 7, 667-675 and 2025-2026.

93 Stewartson, K.
1964 The Theory of Laminar Boundary Layers in Compressible Fluids. Oxford Clarendon Press.

94 Stuart, J.T.
1959 The viscous flow near a stagnation point when external flow has uniform vorticity. J. Aero.Sci. 26, 124-125.

95 Tani, I.
1949 J. Japan Soc. Mech. Engrs. 52, 476.
1954 J. Japan Soc. Mech. Engrs. 57-596.

96 Tricomi, F.G.
1950 Sulla funzione gamma in completa Annali. Mat.pura. appl. 31, 263.

97 Van Driest, E.R.
1959 Convective heat transfer in gases. In: Turbulent Flows and Heat Transfer (High Speed Aerodynamics and Jet Propulsion Vol. 5) edited by C.C.Lin, p. 239-428. Princeton University Press.

98 Van Dyke, M.
1961 Second order compressible boundary layer theory with application to blunt bodies in hypersonic flow. SUDAER NO 112, AFOSR TN-61-1270; also in Hypersonic Flow Research (ed: F.R.Riddell) pp. 37-36, Academic Press, New York.

99 Van Dyke, M.
1962a Higher approximations in boundary layer theory. Part I: General analysis, J. Fluid Mech. 14, 161-177.

100 Van Dyke, M.
1962b Higher approximations in boundary layer theory. Part II Application to leading edges. J.Fluid Mech.14, 481-495.

101 Van Dyke, M.
1962c A review and extension of higher order hyper. boundary layer theory. In: Rarified Gas Dynamics, Proceedings of the Third International Symposium, Paris. Academic Press, New York.

102 Van Dyke, M.
1964 Higher approximations in boundary layer theory. Part III. Parabola in uniform stream. J.Fluid Mech.19, 145-159.

103 Van Dyke, M.
1964 Perturbation Methods in Fluid Mechanics. Academic Press, London.

104 Vasantha, S. and Narasimha, R.
1969 The compressible boundary layer on a flat plate. J. Aero. Soc. India (to be published).

105 Yasuhara, M.
1956 On hypersonic viscous flow past slender bodies of revolution. J. Phys. Soc. Japan 8, 878-886.

106 Yen, K.T. and Toba, K.
1961 A Theory of the two dimensional laminar boundary layer over a curved surface. J. Aero. Sc. 28, 877-884.

107 Yen, K.T. and Toba, K.
1962 Reply to comments on a theory of the two dimensional laminar boundary layer over a curved surface. J. Aero.Sc. 29, 367.

108 Young A.D.
1948 Note on the velocity and temperature distributions attained with suction on a flat plate of infinite extent in compressible flow. Quart. J. Mech. Appl. Math. 1, 70-75.

109 Young, A.D.
1953 Boundary layers. In: Modern Developments in Fluid Dynamics (High Speed Flow) edited by L. Howarth, p.375-362. Oxford University Press.

APPENDIX A

EVALUATION OF INTEGRALS

In Chapers 3 and 4 we have used the following relations

1. $$I_1 = \int_0^z t^n \exp(-at^r/r!)\, dt = \frac{1}{r}\left(\frac{r!}{a}\right)^{\frac{n+1}{r}} \gamma\left(\frac{n+1}{r}, \frac{a}{r!} z^r\right) \quad (A.1)$$

where $\gamma(\alpha, x) = \int_0^x e^{-t}\, t^{\alpha-1}\, dt$

is the incomplete gamma function.

To show the equality of this relation (A.1) we introduce a new variable $y(t) = at^r/r!$ in I_1 to obtain

$$I_1 = \frac{1}{r}\left(\frac{r!}{a}\right)^{\frac{n+1}{r}} \int_0^{y(z)} y^{\frac{n+1-r}{r}} e^{-y}\, dy$$

$$= \frac{1}{r}\left(\frac{r!}{a}\right)^{\frac{n+1}{r}} \gamma\left(\frac{n+1}{r}, \frac{a}{r!} z^r\right).$$

2. $$I_2 = \int_0^\infty z^m \exp(-Pz) \int_0^z t^n \exp(Pt-at)dt \quad dz \quad (A.2)$$

$$= \frac{\Gamma(m+n+2)}{n+1} \; {}_2F_1(m+1, m+n+2, n+2, 1-a/P)P^{-(m+n+2)}$$

where ${}_2F_1(a,b,c,d)$ is the well-known hypergeometric function defined by

$${}_2F_1(a,b,c,x) = \sum_{n=0}^{\infty} (a)_n (b)_n\, x^n / \left[(c)_n\, n!\right]$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ and $\Gamma(\alpha) = \gamma(\alpha, \infty)$ is the complete gamma function.

To prove the relation in (A.2) we first note that the integral in curly brackets is related to incomplete gamma function and with the help of (A.1) can be written as

$$\int_0^z t^n \exp\,(Pt-at)\,dt = (a-P)^{-m-1}\ \gamma(n+1,\ az-Pz)$$

Making use of the relation (see, Erdelyi 1954)

$$\int_0^\infty x^{c-1}\ e^{-\beta x}\,\gamma(d,\ \alpha x)\ dx = \frac{\alpha^d\ \Gamma(c+d)}{d(\alpha+\beta)^{c+d}}\ {}_2F_1(1, c+d, d+1,\ \frac{\alpha}{\alpha+\beta})$$

(valid when real part of $\alpha+\beta$, β, $c+d$ are positive) the integral in (a.2) become

$$I_2 = \frac{\Gamma(m+n+2)}{n+1}\ {}_2F_1\ (1,\ m+n+2,\ n+2,\ 1-P/a)\ a^{-(m+n+2)}$$

Further using the recurrence relation (see, Sneddon 1961)

$${}_2F_1\ (\alpha,\ \beta,\ d,\ 1-x) = x^{-\beta}\ {}_2F_1\ (d-\alpha, \beta,\ d,\ 1-1/x)$$

the above integral I_2 become

$$I_2 = \frac{\Gamma(m+n+2)}{n+1}\ {}_2F_1\ (n+1,\ m+n+2,\ n+2,\ 1-a/P)\ P^{-(m+n+2)}$$

3. $$I_3 = \int_0^x z^m \exp(-az^r/r!)\ \int_0^z t^n \exp\,(at^r/r!)\ dt\ \ dz \qquad (A.3)$$

$$= \frac{1}{r(n+1)}\ (r!/a)^{\frac{m+n+2}{r}}\ N_{\frac{m+n+2-r}{r}}\left(\frac{n+1}{r},\ \frac{a}{r!}\right)$$

where $$N_m(\alpha, x) = \int_0^x t^m\ N(\alpha, t)\ dt$$

$$N(\alpha,\ x) = \alpha \int_0^1 e^{-xt}(1-t)^{\alpha-1}\ dt.$$

Here $N(\alpha,\ x)$ is a certain related gamma function studied by Tricomi (1950) and Narasimha (1964).

The integral with in curly brackets is related to incomplete gamma function with a negative argument (See, Tricomi 1959) and using a procedure similar to in (A.1) we get

$$\int_0^z t^n \exp\,(at^r/r!)\ dr = \frac{1}{r}\,(r!/a)^{\frac{r+1}{n}}\ \gamma_1\left(\frac{n+1}{r}\,,\ \frac{az^r}{r!}\right) \qquad (A.4)$$

where $\gamma_1(\alpha,\ x) = \int_0^x e^t\, t^{\alpha-1}\, dt$. This function is more conveniently studied in terms of a related function $N(\alpha,\ x)$ defined by

$$\gamma_1(\alpha,\ x) = \frac{1}{\alpha}\, e^x\, x^\alpha\, N(\alpha,\ x) \qquad (A.5)$$

or $N(\alpha,\ x) = \alpha e^{-x}\, x^{-\alpha}\ \gamma_1(\alpha, x) = \alpha \int_0^1 e^{-xt}\,(1-t)^{\alpha-1}\, dt.$

By means of (A.4) and (A.5) the integral I_3 can be written

$$I_3 = \frac{1}{r(n+1)}\left(\frac{r!}{a}\right)^{\frac{m+n+2}{r!}} \int_0^{y=\frac{a}{r!}x^r} y^{\frac{m+n+2-r}{r}}\ N\left(\frac{n+1}{r}\ ,y\right) dy$$

$$= \frac{1}{r(n+1)}\,(r!/a)^{\frac{m+n+2}{r}}\ N_{\frac{m+n+2-r}{r}}\left(\frac{n+1}{r},\ \frac{a}{r!}\,x^r\right).$$

Some of the properties of the function $N(\alpha,\ x)$ are (see, Tricomi 1950)

$$0 < N\,(\alpha,\ x) \leq 1, \qquad N(\alpha,\ 0) = 1,$$

$$N(\alpha,\ \infty) = 0, \qquad N(0,\ x) = e^{-x}.$$

Further, it is easily shown (see, Narasimha 1964), that the asymptotic expansions of $N(\alpha,\ x)$ for $x \to 0$ with α fixed is

$$N = 1 - \frac{x}{\alpha+1} + \frac{r^2}{(\alpha+1)(\alpha+2)} - \cdots\cdots$$

and for $x \to \infty$ with α fixed,

$$N = \frac{\alpha}{x} - \frac{\alpha(\alpha-1)}{x^2} + \frac{\alpha(\alpha-1)(\alpha-2)}{x^3} - \cdots\cdot;$$

APPENDIX B

EULERISATION: (E, q) METHOD

To improve the convergence of a slowly convergent or even divergent series various methods are used in the literature which accelerate the convergence. Several interesting examples of such transformations are given by Van Dyke (1964). One of the most widely used being the process called Eulerisation (See, Meksyn 1961). It is found in the present work, however, that although the conventional Eulerisation procedure was not always successful, an extension described by Hardy (1949) and called by him the (E, q) process gave satisfactory results.

Following Hardy (1949) to improve the convergence of a series

$$S_n = \sum_{n=0} a_n \; x^{n+1} \tag{B-1}$$

which is convergent for sufficiently small values of x, by the (E, q) method, we proceed as follows.

Let $\qquad x = y/(1-qy)$ or $y = x/(1+qx)$, $q \neq 0$. $\qquad$ (B-2)

It is easily seen that the substitution of (B-2) in (B-1) gives

$$S_n = \sum_{m=0} a_m^{(q)} \; (y(1+q))^{m+1} \tag{B-3}$$

where

$$a_m^{(q)} = \frac{1}{(q+1)^{m+1}} \sum_{n=0}^{r} \binom{m}{n} q^{m-n} a_n .$$

If we choose x = 1, then (1+q)y = 1, and the series (B-4) reduces to

$$S_n = \sum_{m=0} a_m^{(q)} \tag{B-4}$$

The series (B-4) is said to be the Eulerised form (by q) of $S_n = \sum_{n=0} a_n x^{n+1}$. To Eulerise a given series, this q has to be determined by a ftrial and error procedure (in the conventional Eulerization procedure q is taken as unity). Further, in practice it is found convenient to Eulerise a series leaving out some of the leading terms of the expansion, as Fulerization including in particular (E,q) method, leads to unique results (See, Hardy 1949), the final result does not depend in principle on the number of terms left out. It may be noted that it can be shows (see, Hardy 1949), that if a series is summable (E, q') and q > q', then it is also summable (E, q) to the same sum.

To plot the Eulerised results of low σ series (4.45) for heat transfer in Fig. 4.4 and 4.5 we found that Eulerisation with the following values of q gave rapid convergence,

1. $$D(-0.198838,\sigma) = 0.79788\,\sigma^{1/2} - 1.05017\,\sigma + 3.24851\,\sigma^{3/2} - 6.93062\,\sigma^2 + 0(\sigma^{5/2})$$

σ	q	Leading terms left
0.05	1	First only
0.1	1	
0.3	1	
0.6	2	

2. $$D(-0.16,\sigma) = 0.79788\,\sigma^{1/2} - 1.08649\,\sigma + 1.85581\,\sigma^{3/2} - 3{,}07590\,\sigma^2 + 0(\sigma^{5/2})$$

0.1	1	First only
0.5	1	
1.0	2	

3. $$D(0,\sigma) = 0.79788\sigma^{-1/2} - 0.77463\sigma + 1.03212\sigma^{-3/2} - 1.31774\sigma^{2} + O(\sigma^{-5/2})$$

σ	q	Leading terms left
0.3	1	First only
0.5	1	
0.8	1	
1.0	1	
4.0	3	
9.0	4	
16.0	9	

Further, to plot the Eulerised low σ result (4.44) for the recovery factor in Fig. (4.6) the following values of q gave rapid convergence,

$$r_f(\beta=0,\sigma) = 0.9255\,\sigma^{1/2} + 0.1951\,\sigma - 0.1661\sigma^{3/2} + 0.0236\sigma^{2} + O(\sigma^{5/2})$$

0.9	1	First two only
1.0	1	
4.0	2	
9.0	3	First only .

It is well known that the success of the usual Eulerization procedure (q = 1) rests on the fact that the function represented by the series (say in x) possesses a simple pole singularity in the complex x - plane near x = -1 (See, Van Dyke 1964). From (B-2) it appears that (E,q) method is an extension to allow for the presence of the pole near x = -1/q. The fact that in present work the appropriate q to use depends itself on x is perhaps due to the presence of more complicated singularities than simple poles in complex $\sigma^{-1/2}$ plane. The main point is that <u>one</u> can determine a value of q which gives satisfactory convergence, so that the final results so obtained from a series

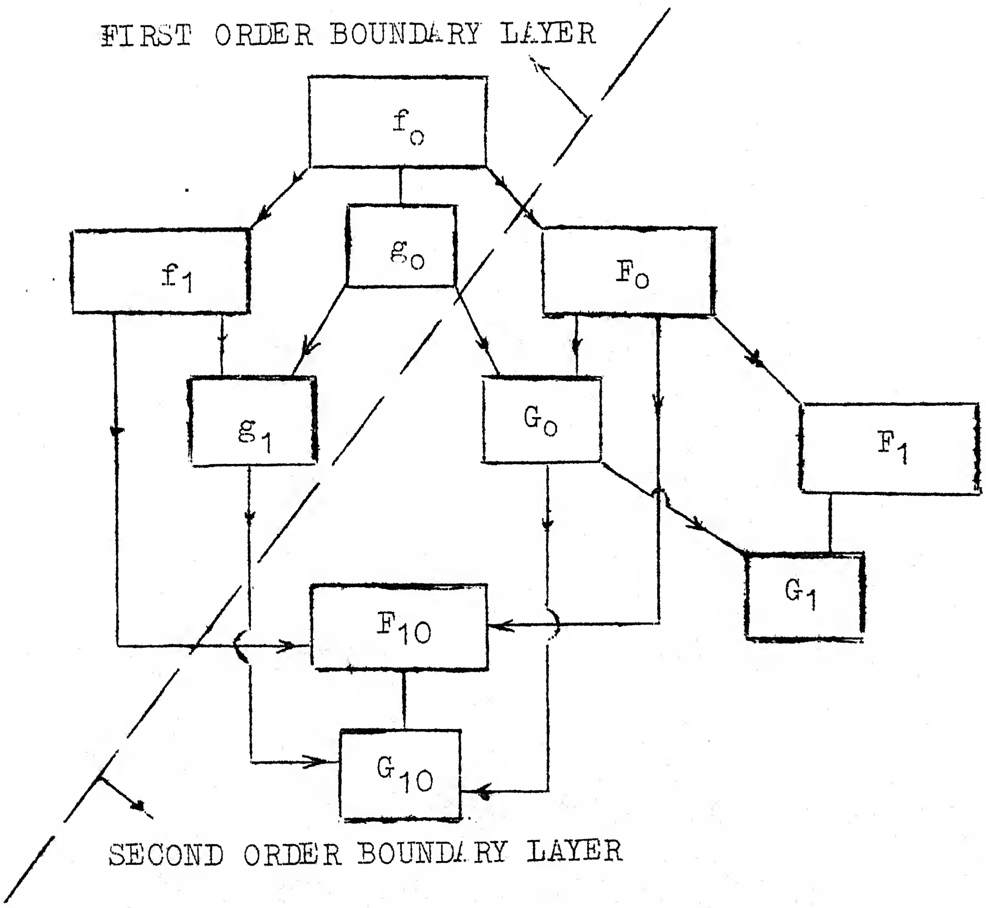
nominally constructed for low σ are excellent for σ so high that they agree with results obtained by considering the opposite limit of large σ.

APPENDIX C

NUMERICAL SOLUTION OF HIGHER ORDER BOUNDARY LAYER PROBLEMS

A general programme to solve, any finite number of total differential equations of the type studied in the present work, numerically by Runge-Kutta method with Gill improvement is developed for IBM 7044 Computer at Indian Institute of Technology, Kanpur. To illustrate the method of solution, let us consider the governing total differential equations for any one of the second order effects, say, transverse curvature. The problem of transverse curvature is defined by the sets of equations (5.55), (5.56) and (5.57) in addition to the corresponding first order boundary layer equations (5.23) and (5.29). Each of individual equation of above mentioned sets constitutes the two point boundary value problem; since for example, for velocity profile two of the boundary conditions are known at $\eta = o$, the third known at $\eta = \infty$. To solve each of such equations it is necessary to guess a condition missing at the boundary $\eta = 0$ and integrate towards $\eta = \infty$ to see if the boundary condition is meet here.The problem for f_o is nonlinear and therefore some what trouble some to handle. However, the solution to this equation is given by various authors mentioned in Chapter 4, the missing value $f_o''(0)$ is taken from Smith (1954) in the present work. Rest of the equations are linear two point boundary value type and may be

integrated easily by some sort of iteration or interpolation process. However, to solve each of these equations separately (after obtaining the solutions to previous ones) is unnecessary complicated and time consuming. We here have adopted a more efficient technique by solving at a time as many equations as possible. From the following chart one may find such number of equations which can be solved at a time



Here

$$f = f_o + \xi \beta_1 f_1 + \cdots$$

$$g = g_o + \xi \beta_1 g_1 + \cdots$$

$$F = k_{ot} F_o + \xi (k_{1t} F_1 + \beta_1 k_{ot} F_{1o}) + \cdots$$

$$G = k_{ot} G_o + \xi (k_{1t} G_1 + \beta_1 k_{ot} G_{1o}) + \cdots .$$

From the chart it is obvious that once the solution for f_o is known (for example by generating the profile from the initial values of $f_o''(0)$ obtained from Smith) we can solve simultaneously three equations g_o, f_1, F_o at a time in the first instance. In second we can solve g_1, G_o, F_1, F_{10} and in the third G_1, G_{10}. To accomplish the above, the programme is to be engineered so as to study either in three different loops or all equations in a loop. The later one is more efficient and useful. To write up such a programme which can handle all equations at a time it was found convenient that the equations be first expressed as a set of first order differential equations. The each set of equations is therefore reduced to a set of five first order equations as follows.

f_o : $X(2,J) = f_o$, $X(3, J) = f_o'$, $X(4, J) = f_o''$

$X(2,0) = 0 = X(3,0)$, $X(3, \infty) = 1$

g_o : $X(5,J) = g_o$, $X(6,J) = g_o'$

$X(5,0) = 1$, $X(5, \infty) = 0$

f_1 : $X(7,J) = f_1$, $X(8\ \ J) = f_1'$, $X(9,J) = f_1''$

$X(7,J) = X(8, 0) = X(7, \infty) = 0$

g_1 : $X(10,J) = g_1$, $X(11, J) = g_1'$

$X(10,0) = 0 = X(10, \infty)$.

F_o : $X(12,J) = F_o$, $X(13, J) = F_o'$, $X(14, J) = F_o''$

$X(12, 0) = 0 = X(13,0)$, $X(13, \infty) = X(1, \infty)$

G_o : $X(15,J) = G_o$, $X(16,J) = G_o'$

$X(15,0) = 0 = X(15, \infty)$

F_1 : $X(17,J) = F_1$, $X(18, J) = F_1'$, $X(19, J) = F_1''$

$X(17,0) = 0 = X(18, 0)$, $X(18, \infty) = X(1, \infty)$

G_1 $X(20,J) = G_1$, $X(21, J) = G_1'$

$X(20, 0) = 0 = X(20, \infty)$

F_{10} : $X(22,J) = F_{10}$, $X(23,J) = F_{10}'$, $X(24, J) = F_{10}''$

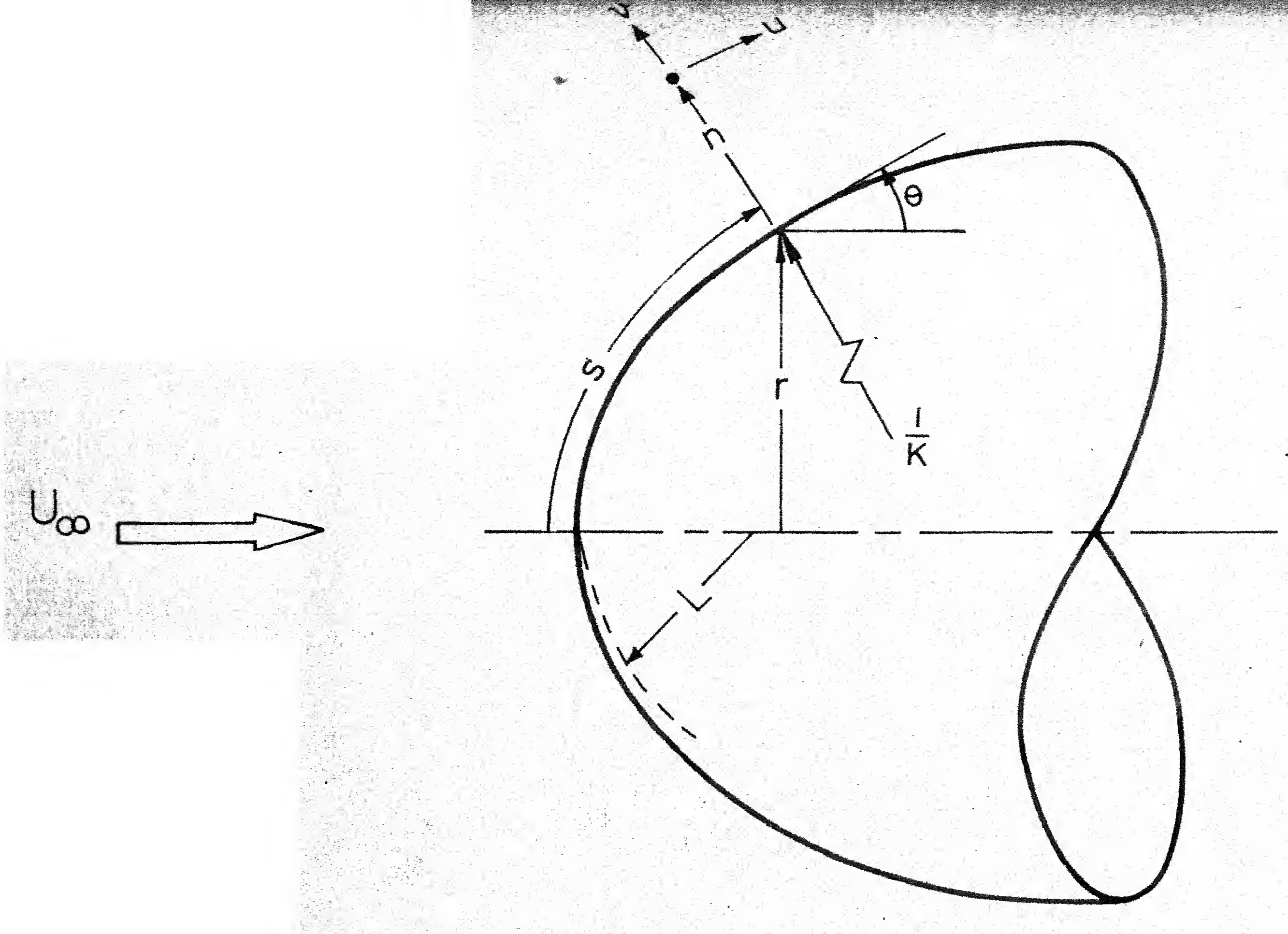
$X(22, 0) = X(23, 0) = X(23, \infty) = 0$

G_{10} $X(25,J) = G_{10}$, $X(26, J) = G_{10}'$

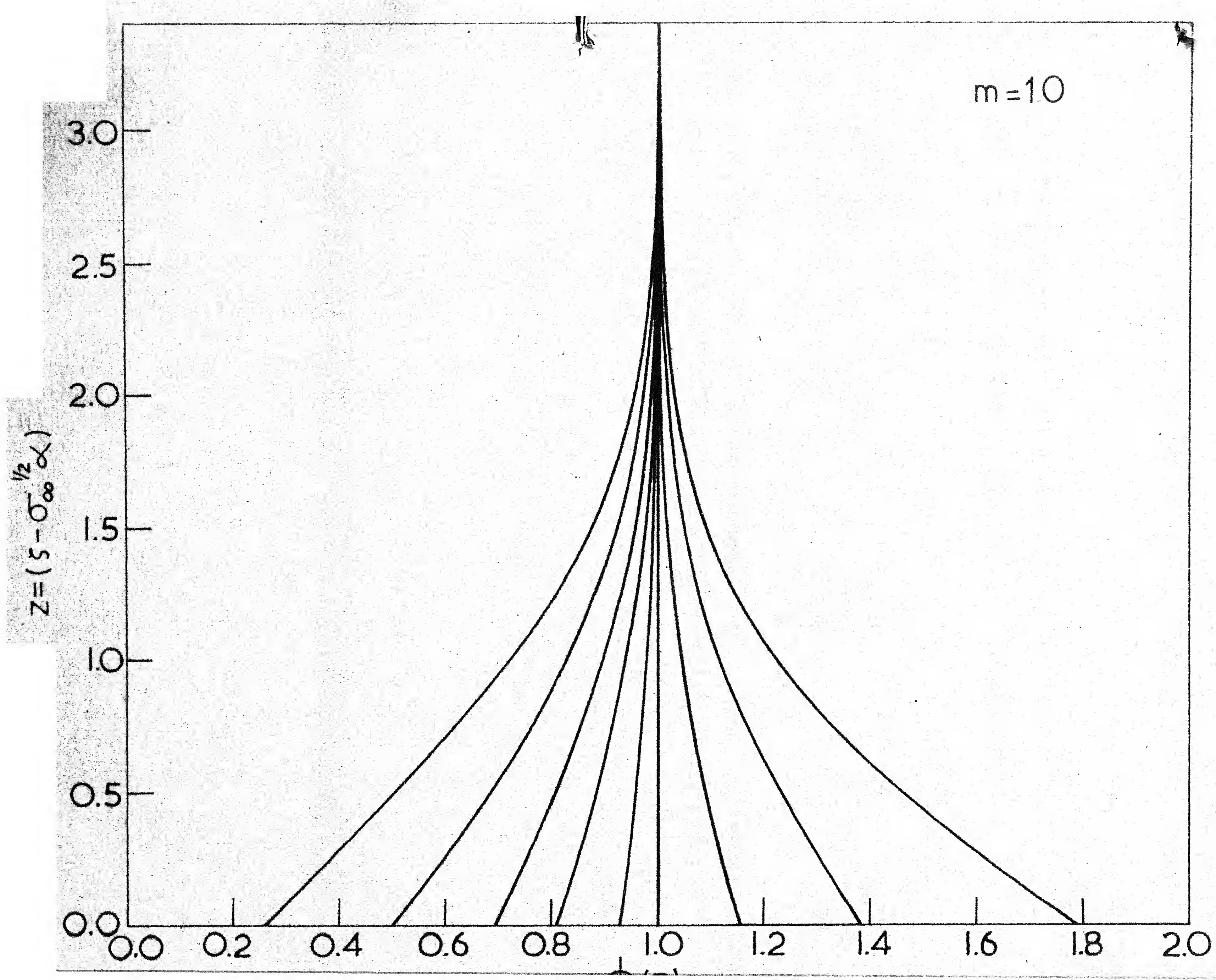
$X(25, 0) = 0 = X(26, \infty)$,

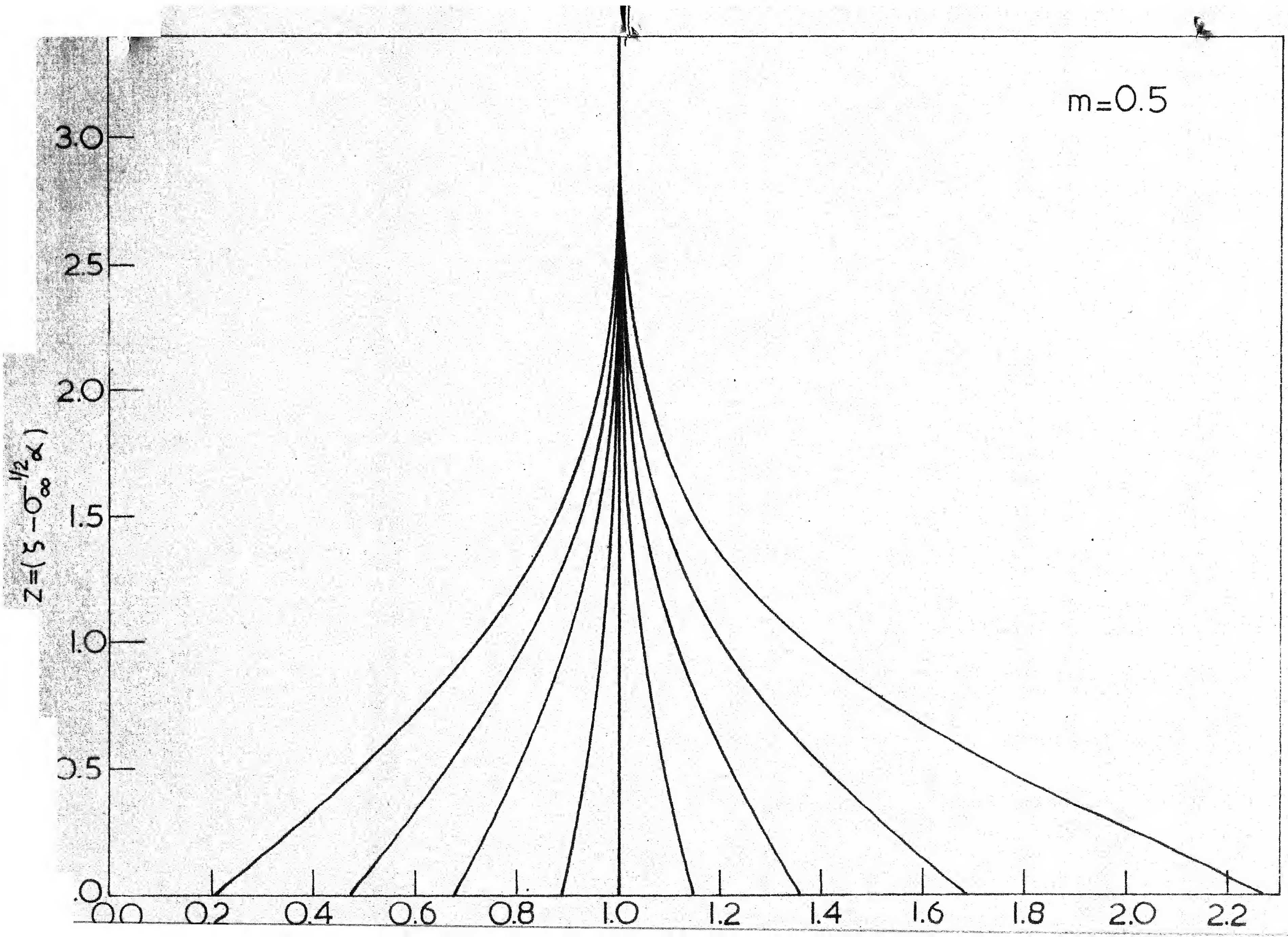
where $X(1, J) =$.

A programme was developed so as to handle all 25 first order equations in one loop. Assuming suitably the missing tboundary conditions, the equations are integrated numerically by Runge-Kutta-Gill method from $\eta = 0$ to $\eta = \eta_o$ (where η_o was chosen to be between 5 to 9 depending upon β_o and σ) with a step size $\Delta\eta = 0.05$. After completion of first integration, the second integration is performed by giving a small perturbation to all the guessed quantities in the first integration. The two solutions were interpolated linearly so as to satisfy the boundary condition at $\eta = \eta_o$. The procedure of integration is repeated by setting up an iterative loop. The initial values for each integration were corrected with the newly determined final values at the infinity. It was found that after few trials the procedure converged to the correct boundary conditions for the entire system.



ig 2.1_ Coordinate system


m = 1.0
$z = (\varsigma - \sigma_\infty^{1/2} \alpha)$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0


m=0.5
$z = (\xi - \sigma_\infty^{-1/2}\alpha)$
0
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2

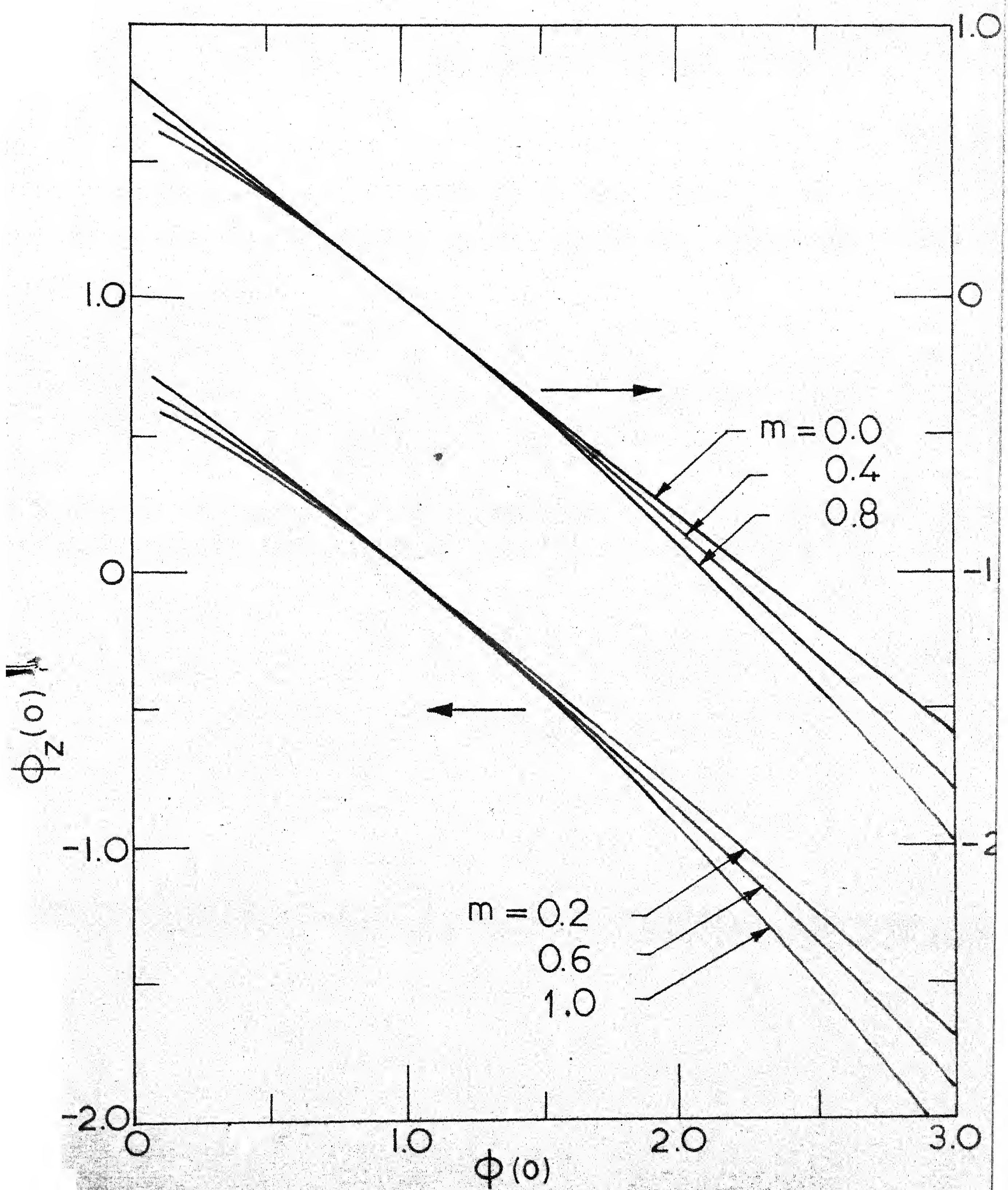
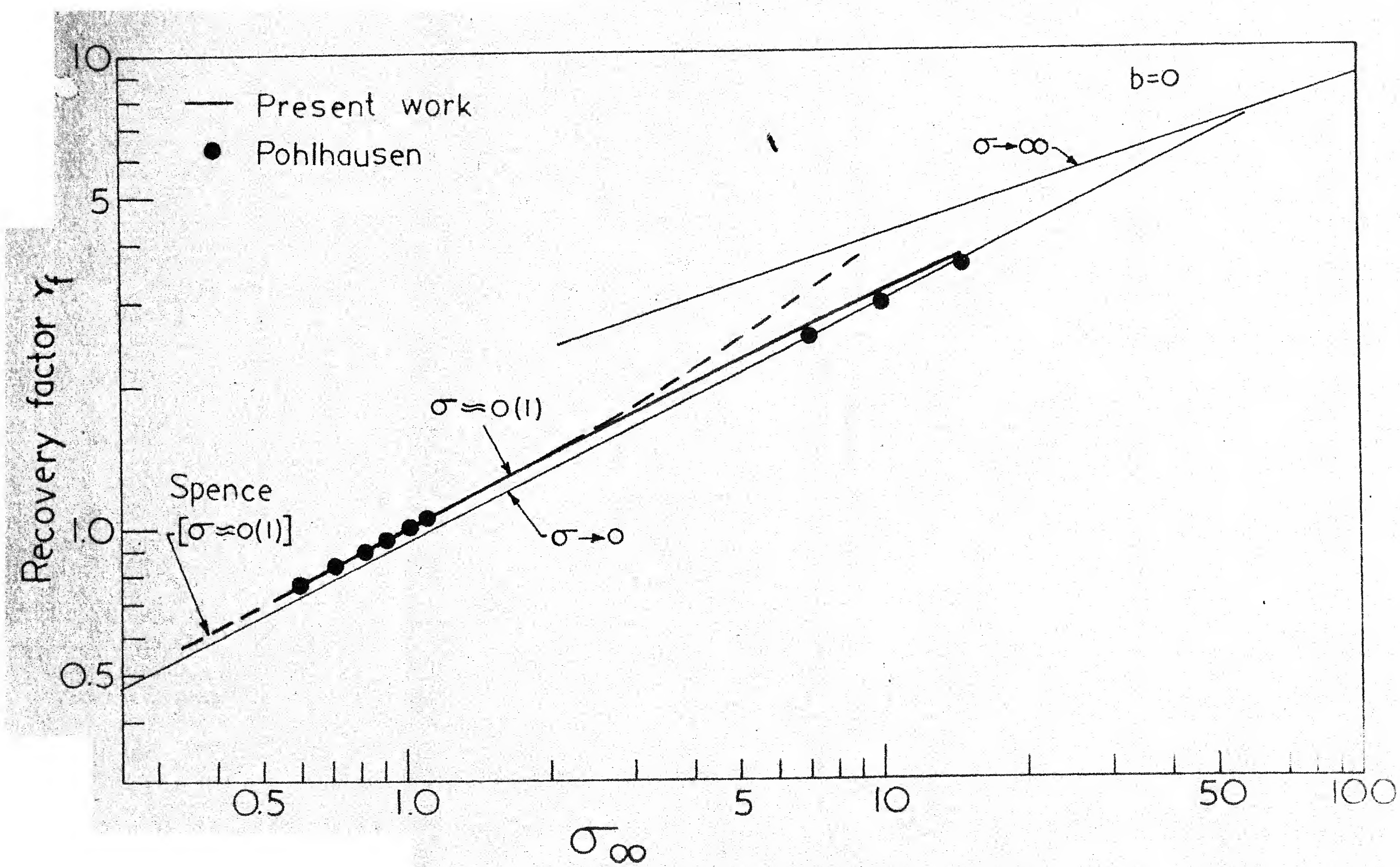
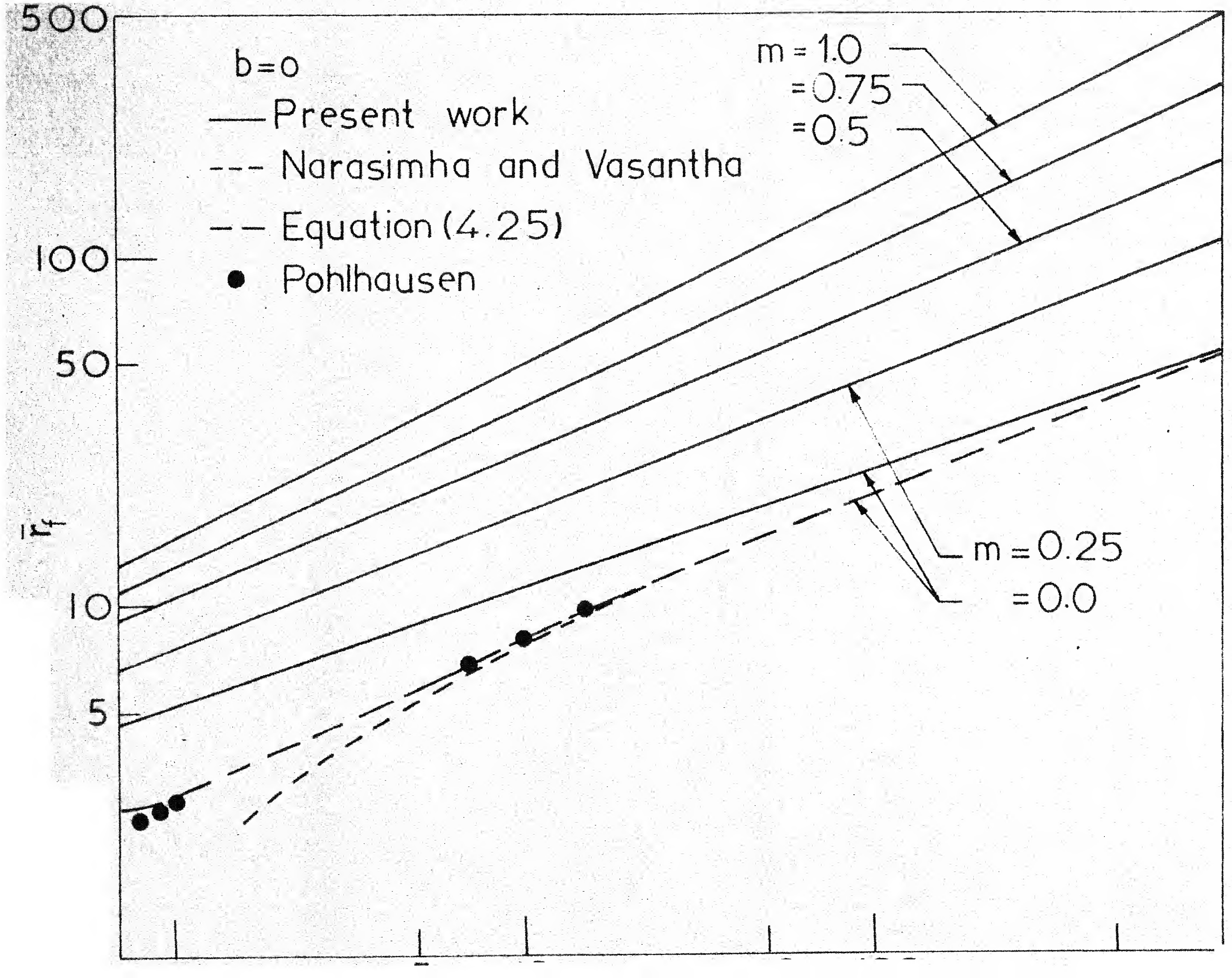


Fig.3.3 Initial values of the solution of outer equation (low Prandtl number flows)



Present work
Pohlhausen
b=0
σ→∞
σ≈0(1)
σ→0
Spence
[σ≈0(1)]
Recovery factor r_f
σ_∞
10
5
1.0
0.5
0.5
1.0
5
10
50
100



500
100
50
10
5
$\bar{r}_t$
b=o
Present work
Narasimha and Vasantha
Equation (4.25)
Pohlhausen
m = 1.0
= 0.75
= 0.5
m = 0.25
= 0.0

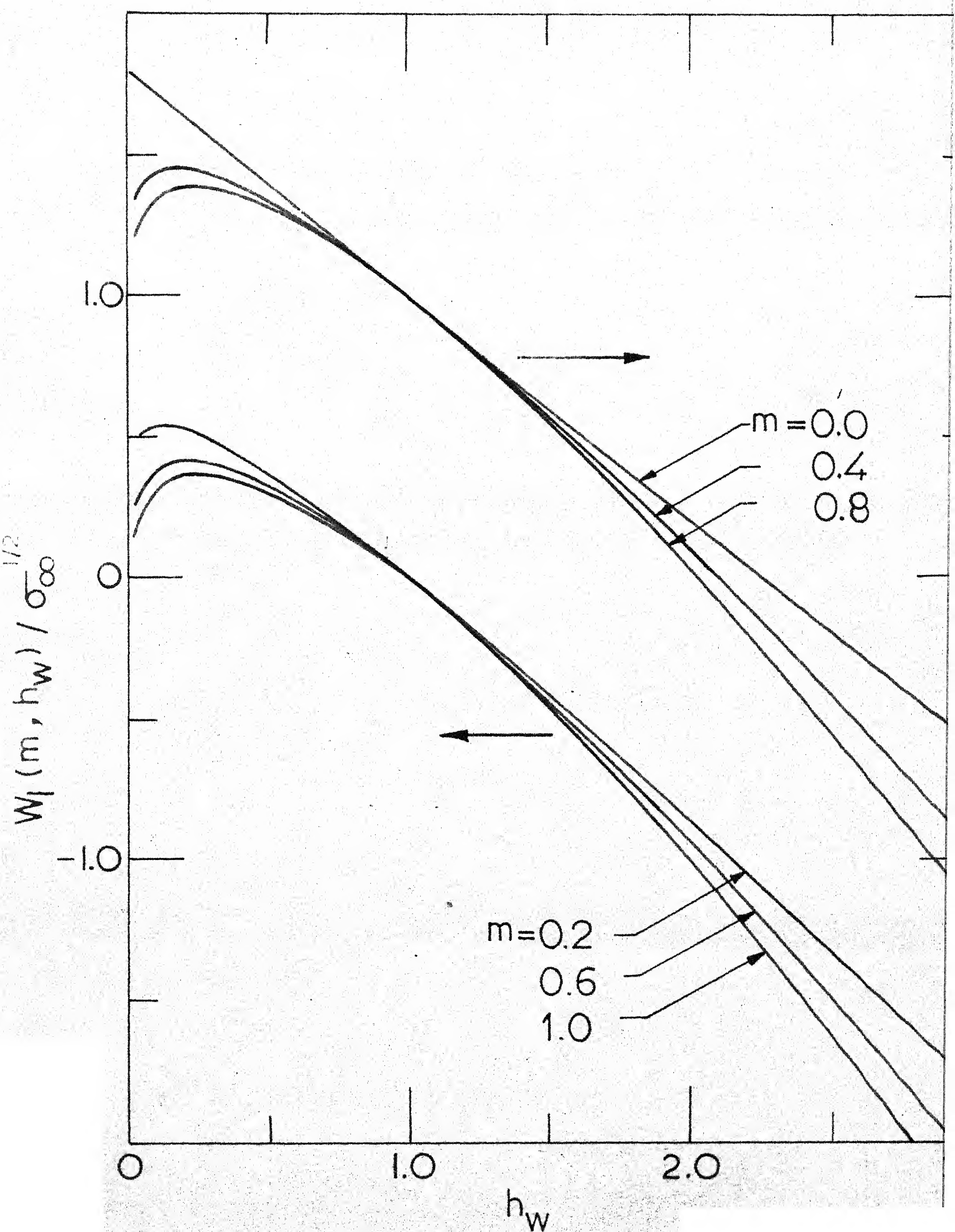


Fig. 3.6 _ $W_I(m, h_W)$ vs h_W

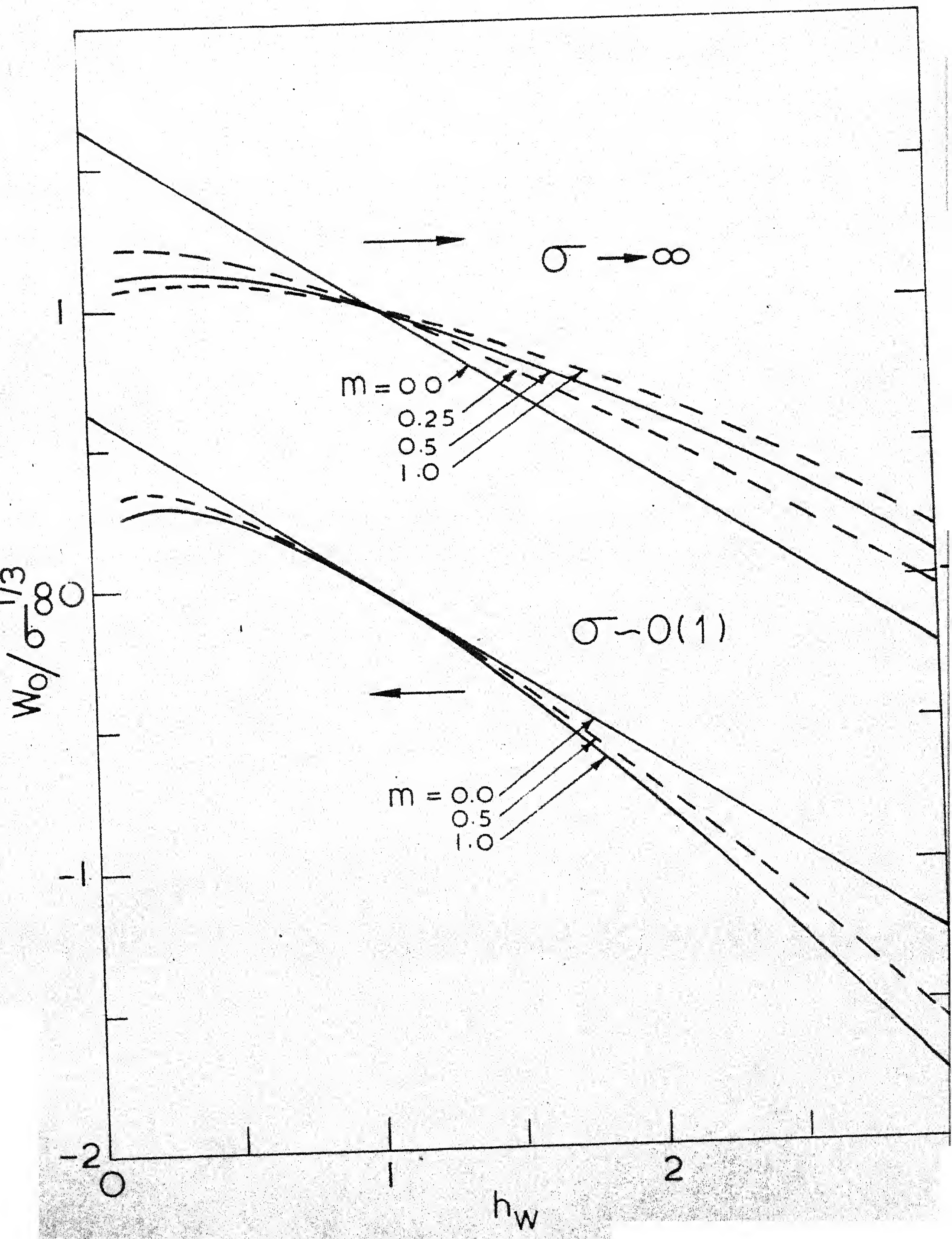


Fig. 3.7 W_0 and W_h vs h_w

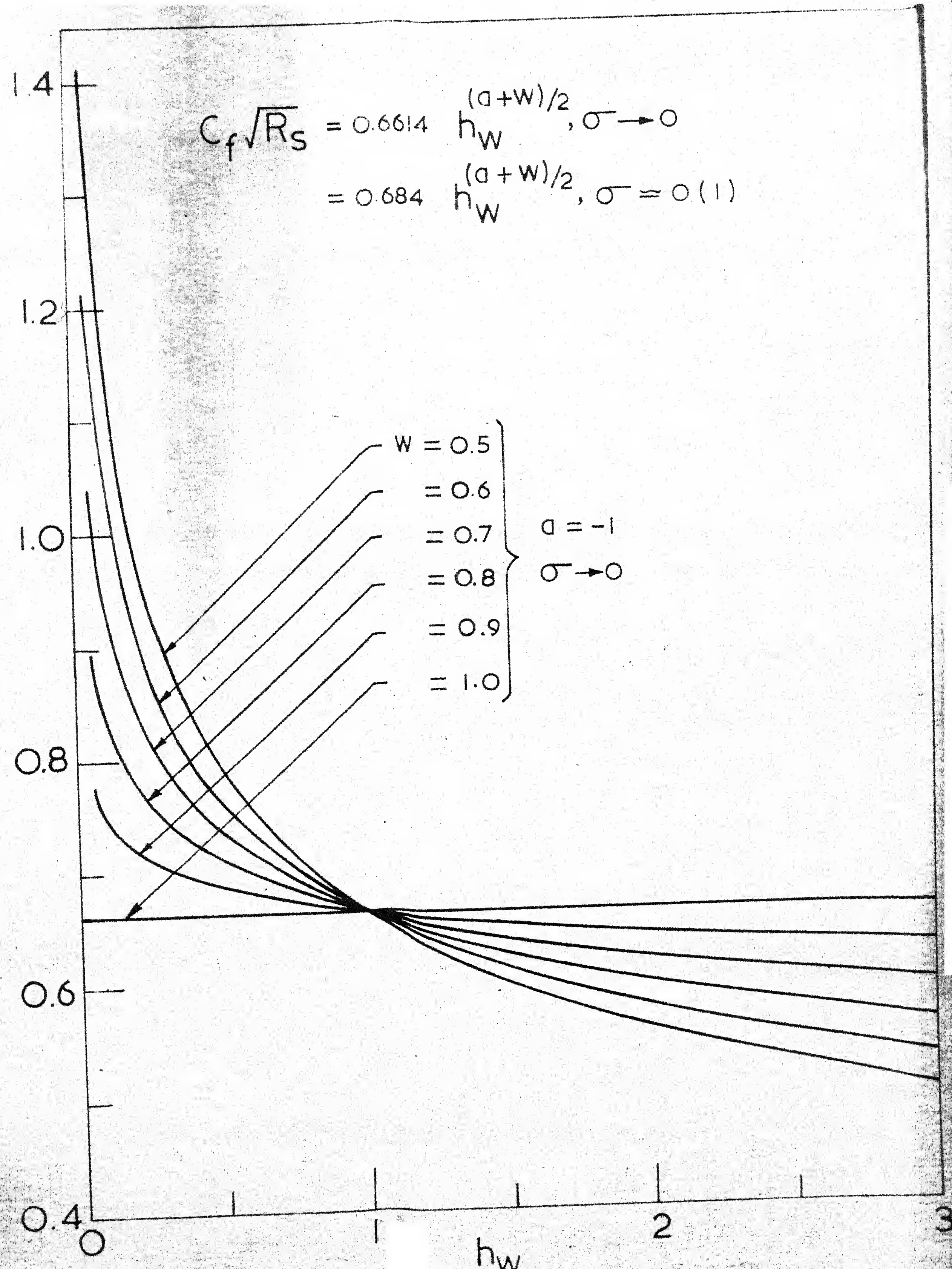


Fig.3.8_Variation of skin friction with thermal properties.

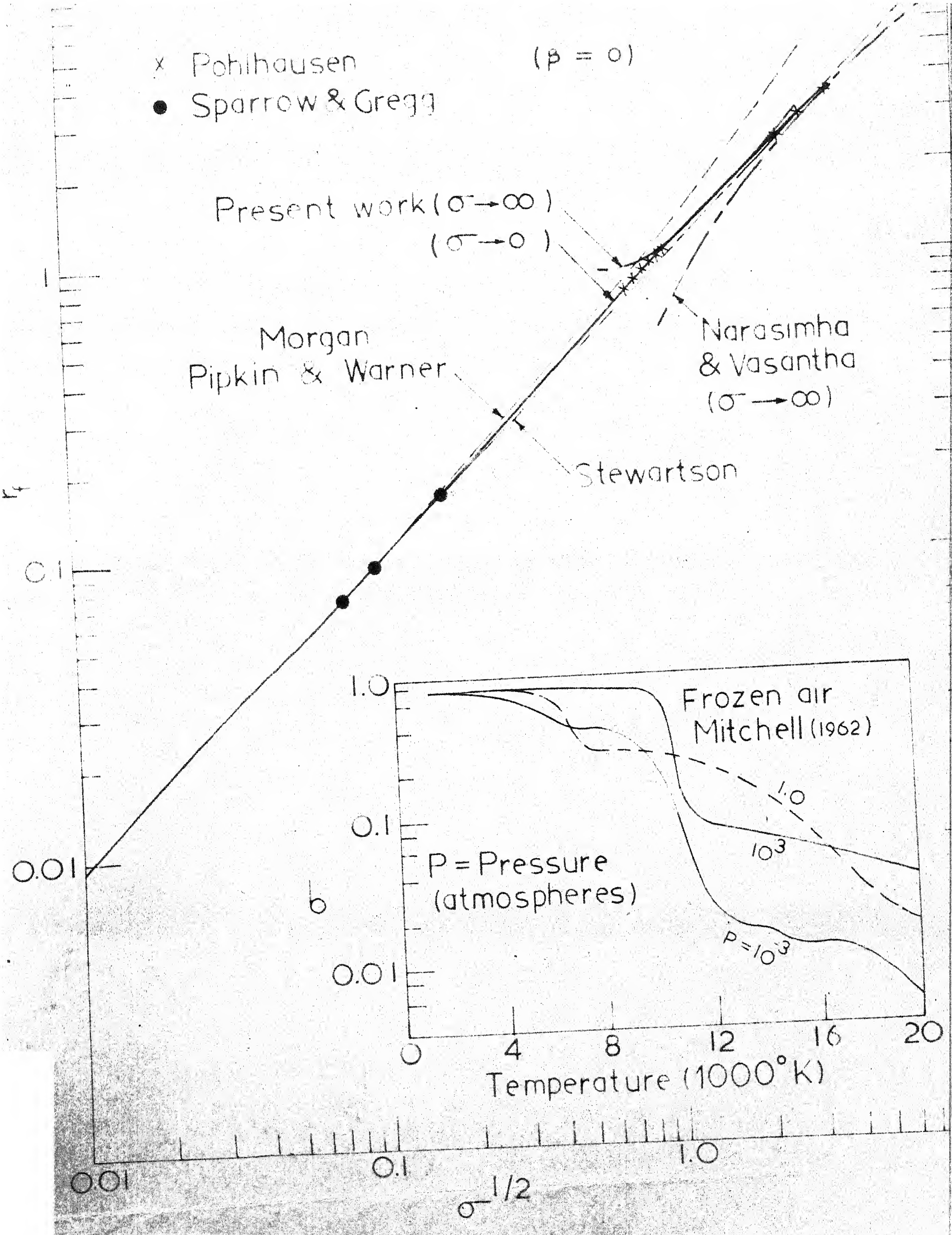


Fig. 4.6 Comparison of the recovery factor results

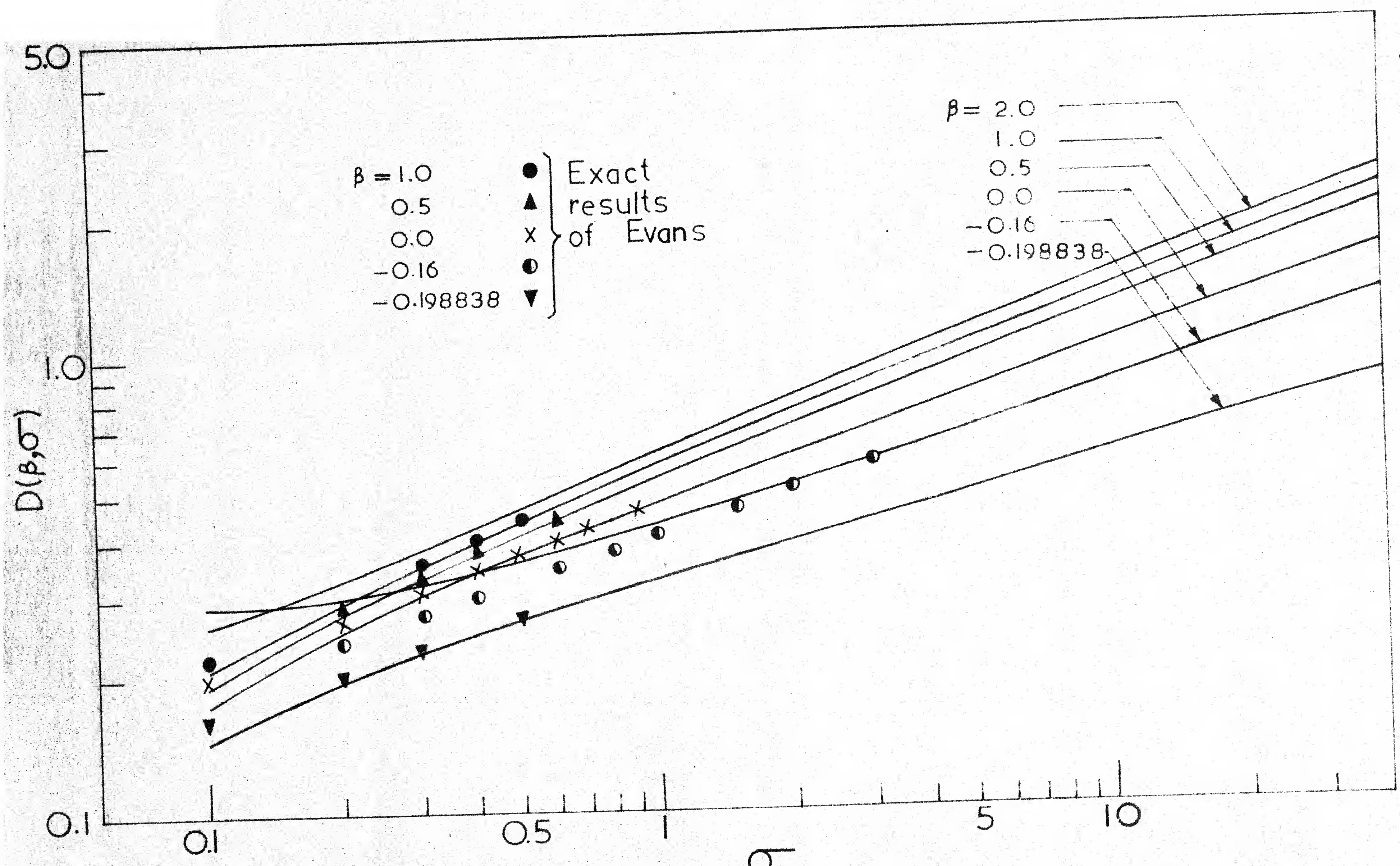


Fig. 4.1_Function D(β , σ) Vs β for high Prandtl number flows

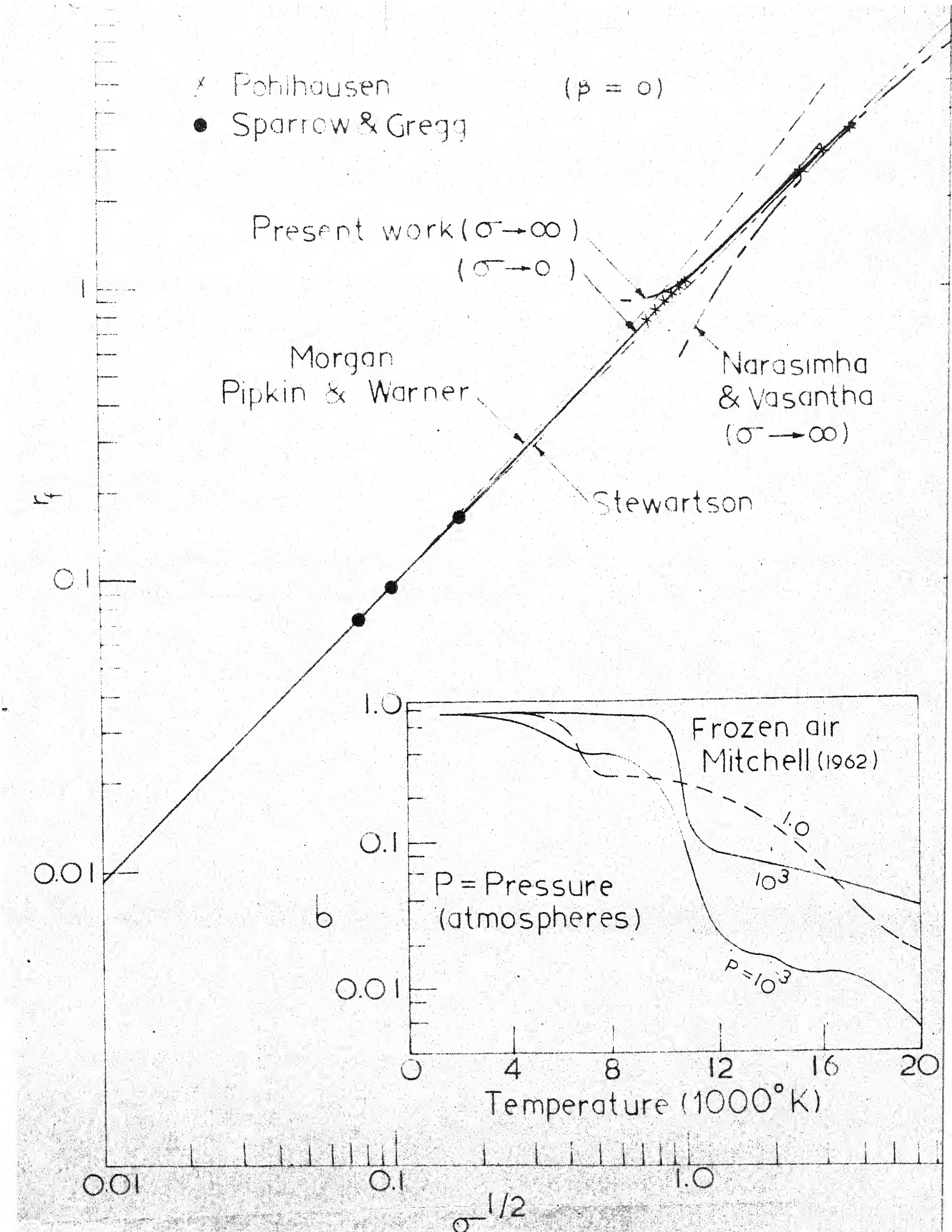


Fig. 4.6_Comparison of the recovery factor results

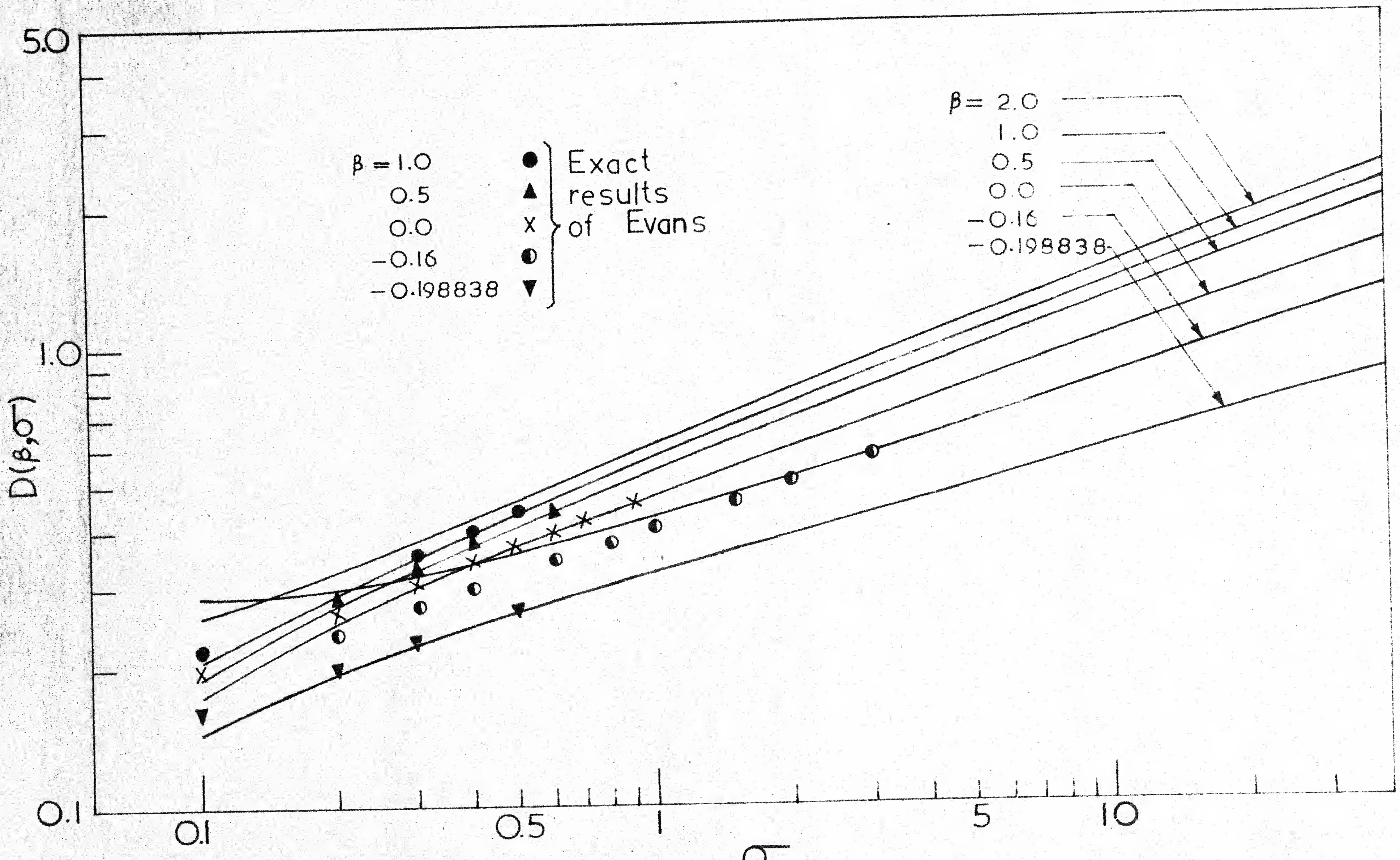


Fig. 4.1_Function $D(\beta, \sigma)$ Vs β for high Prandtl number flows

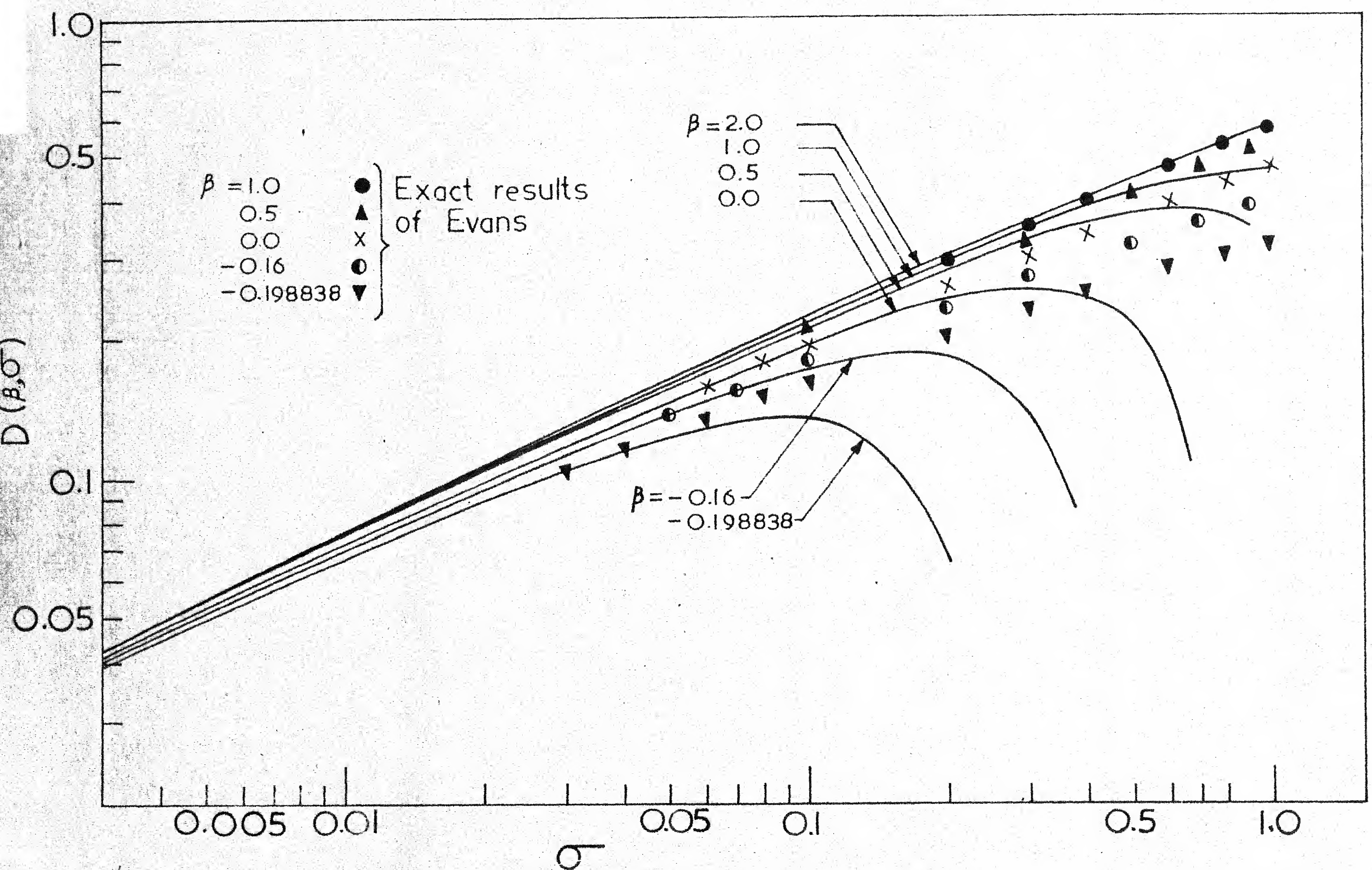


Fig. 4.2_Function $D(\beta, \sigma)$ Vs β for low Prandtl number flows

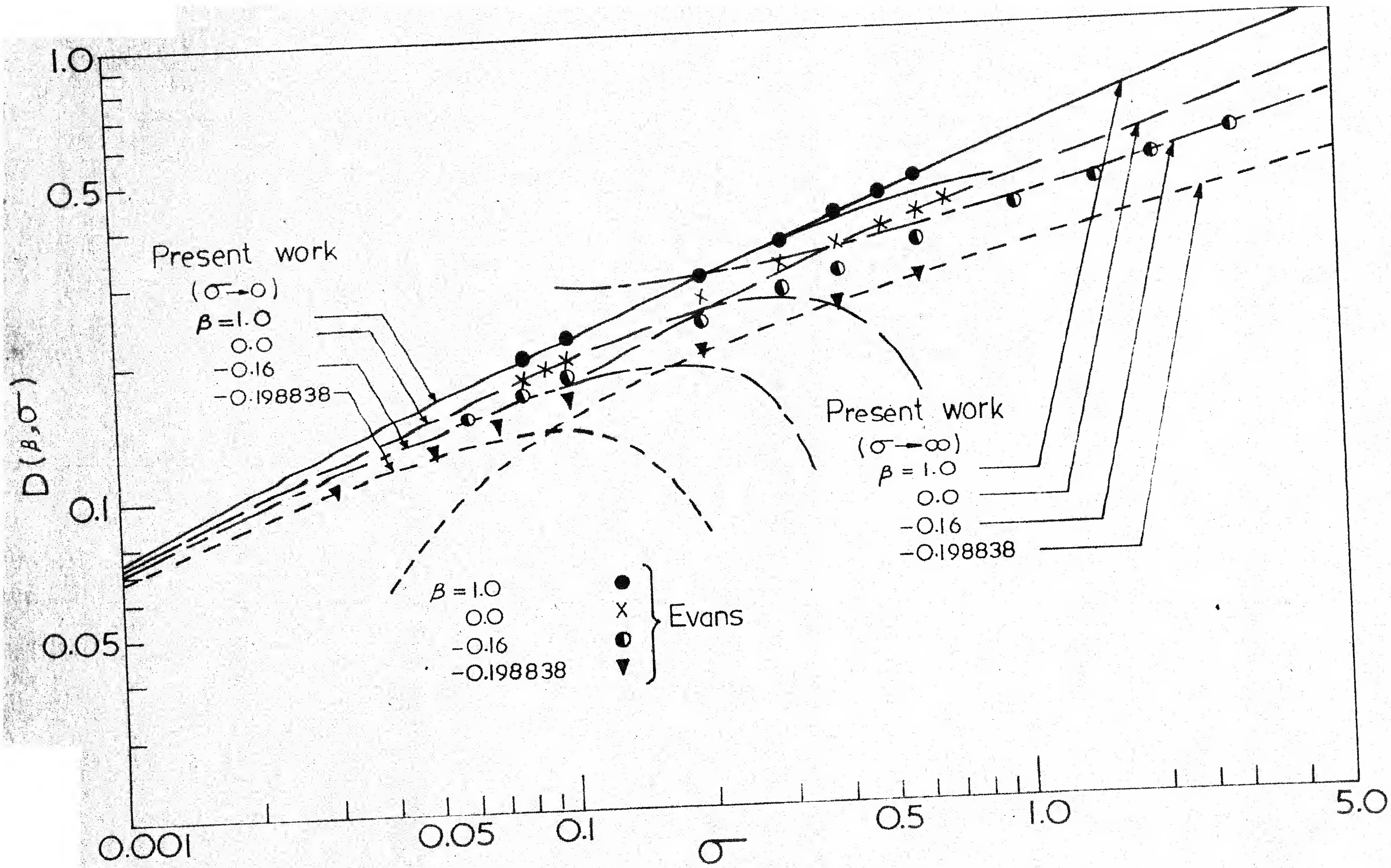


Fig. 4.3_Comparison of $D(\beta,\sigma)$ for low and high Prandtl numbers in the intermediate region

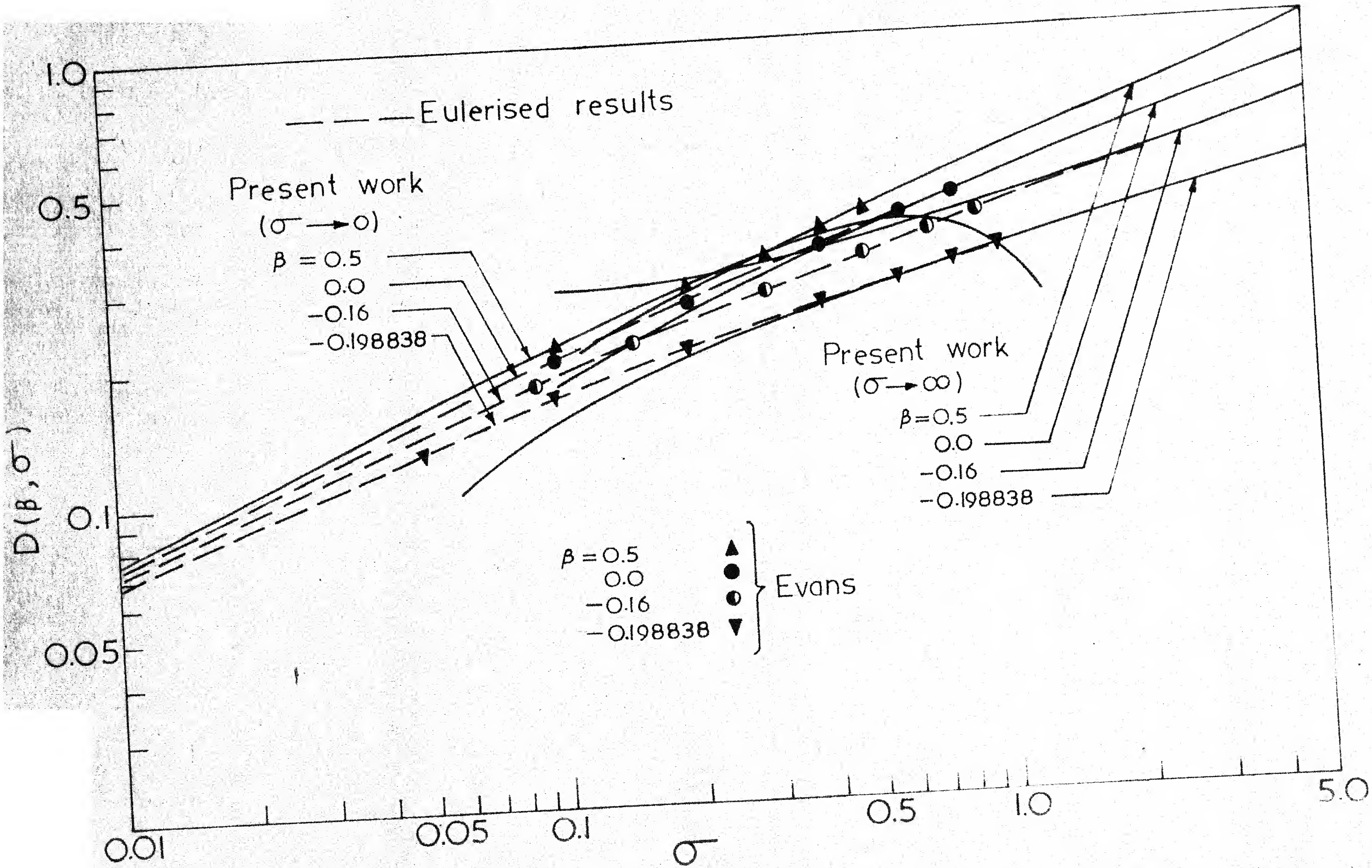
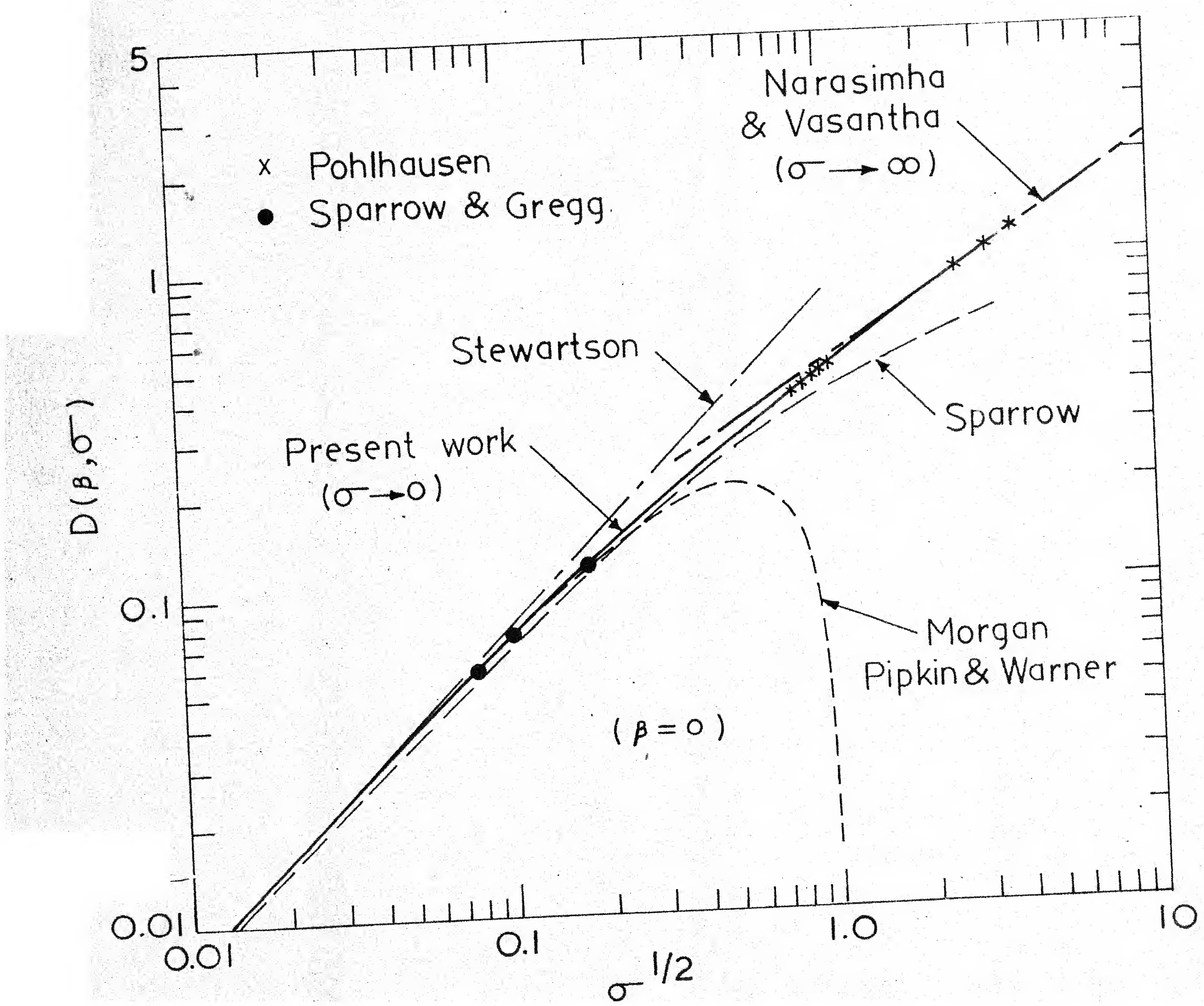


Fig.4.4_Comparison of Eulerised results for $D(\beta,\sigma)$ in the intermediate range



5
1
0.1
0.01
0.01
0.1
1.0
10
D(β,σ)
σ 1/2
x Pohlhausen
● Sparrow & Gregg
Narasimha
& Vasantha
(σ → ∞)
Stewartson
Present work
(σ → 0)
Sparrow
Morgan
Pipkin & Warner
(β = 0)

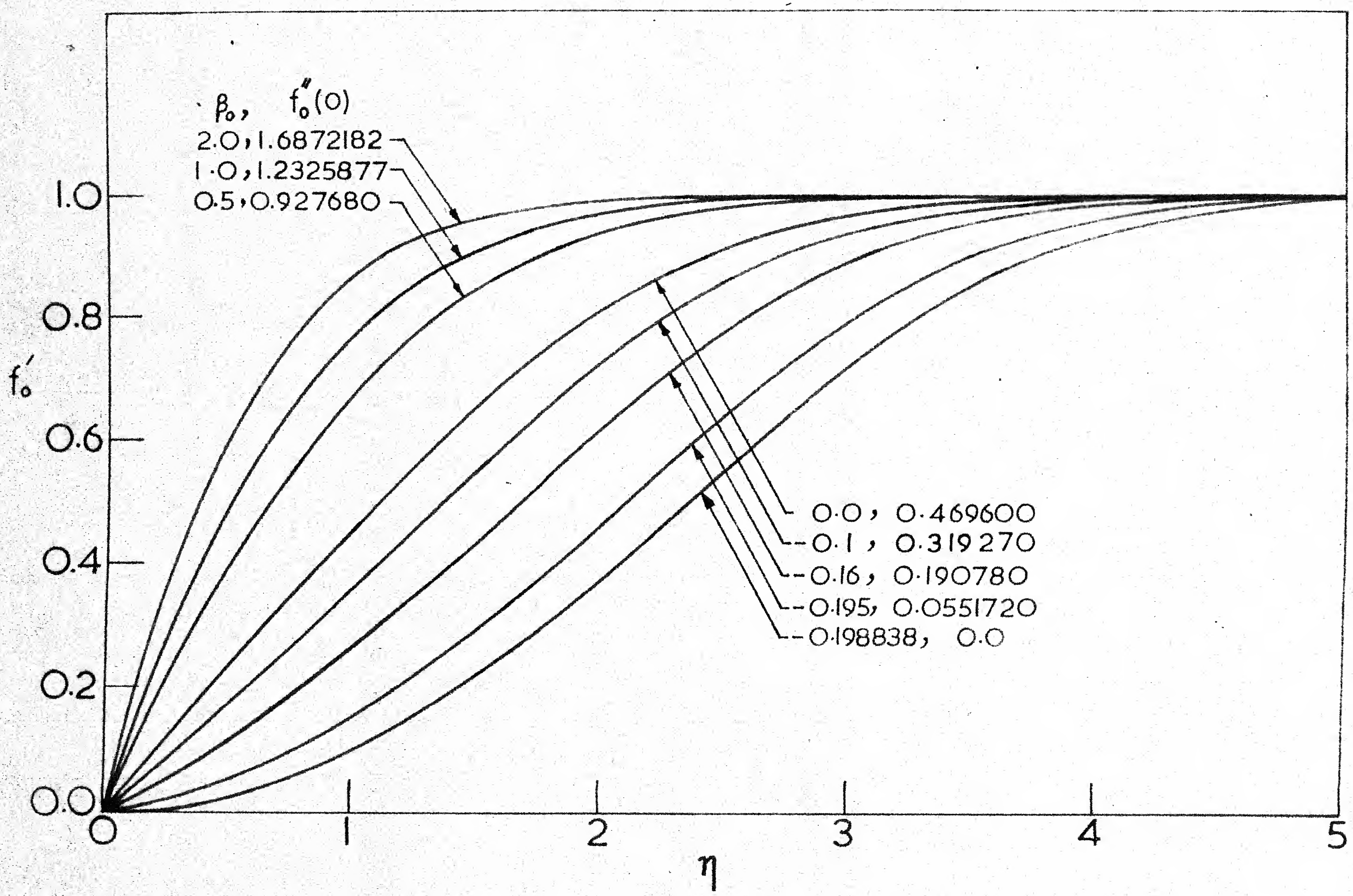


Fig.5.1_Basic velocity profiles (Eqn. 5.23a)

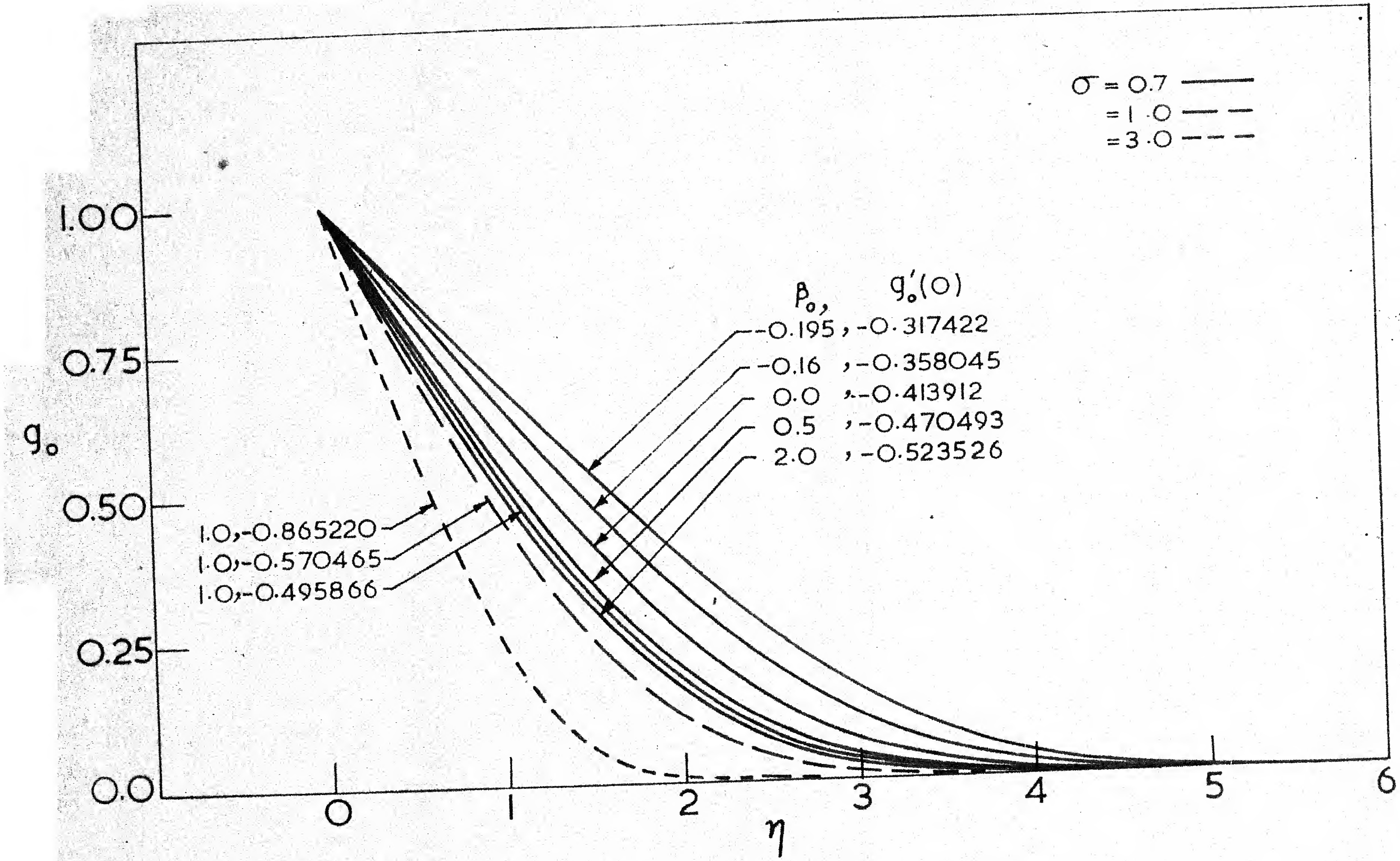


Fig. 5.2 – Basic temperature profiles (Eqn. 5.23b)

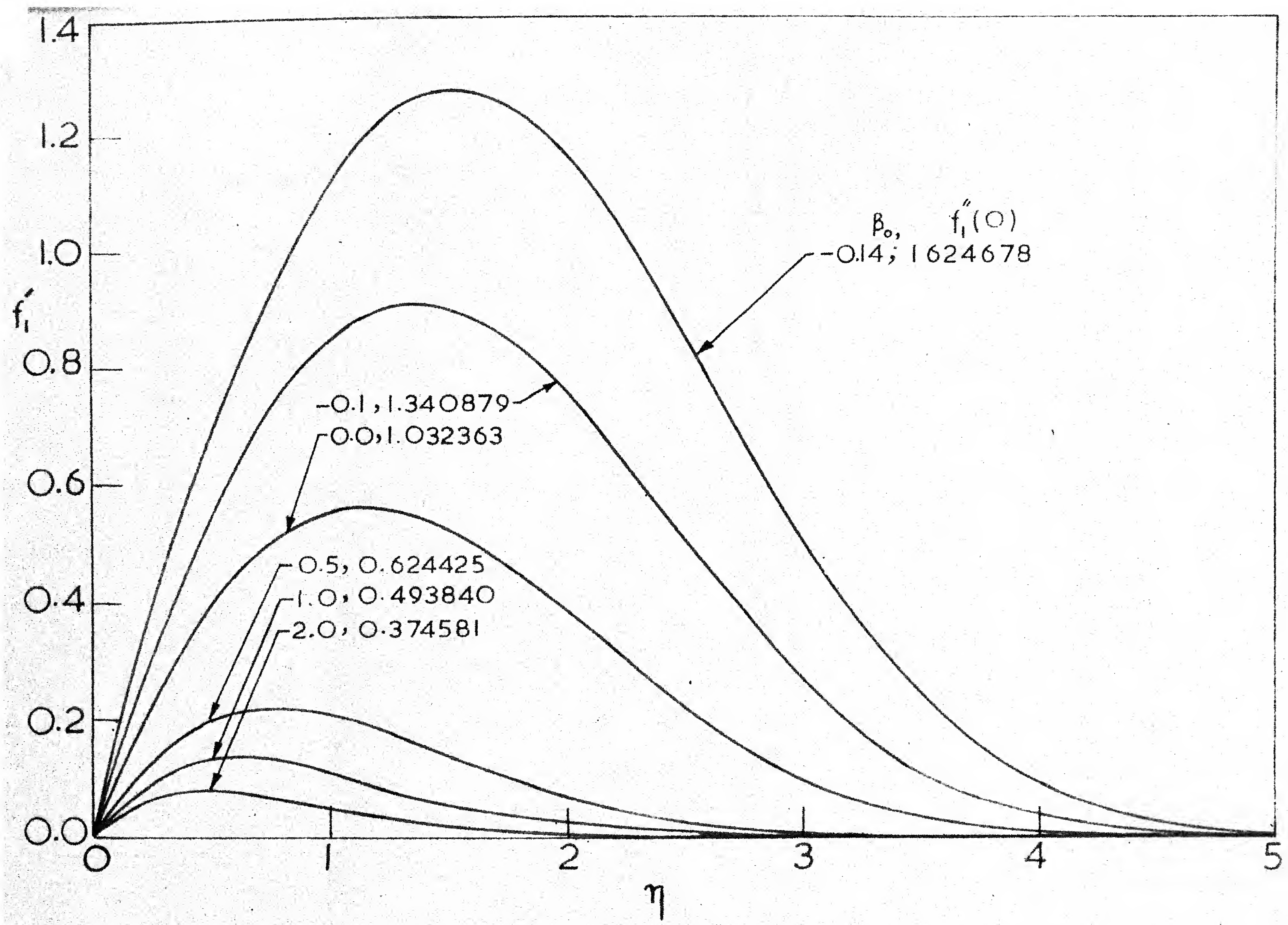
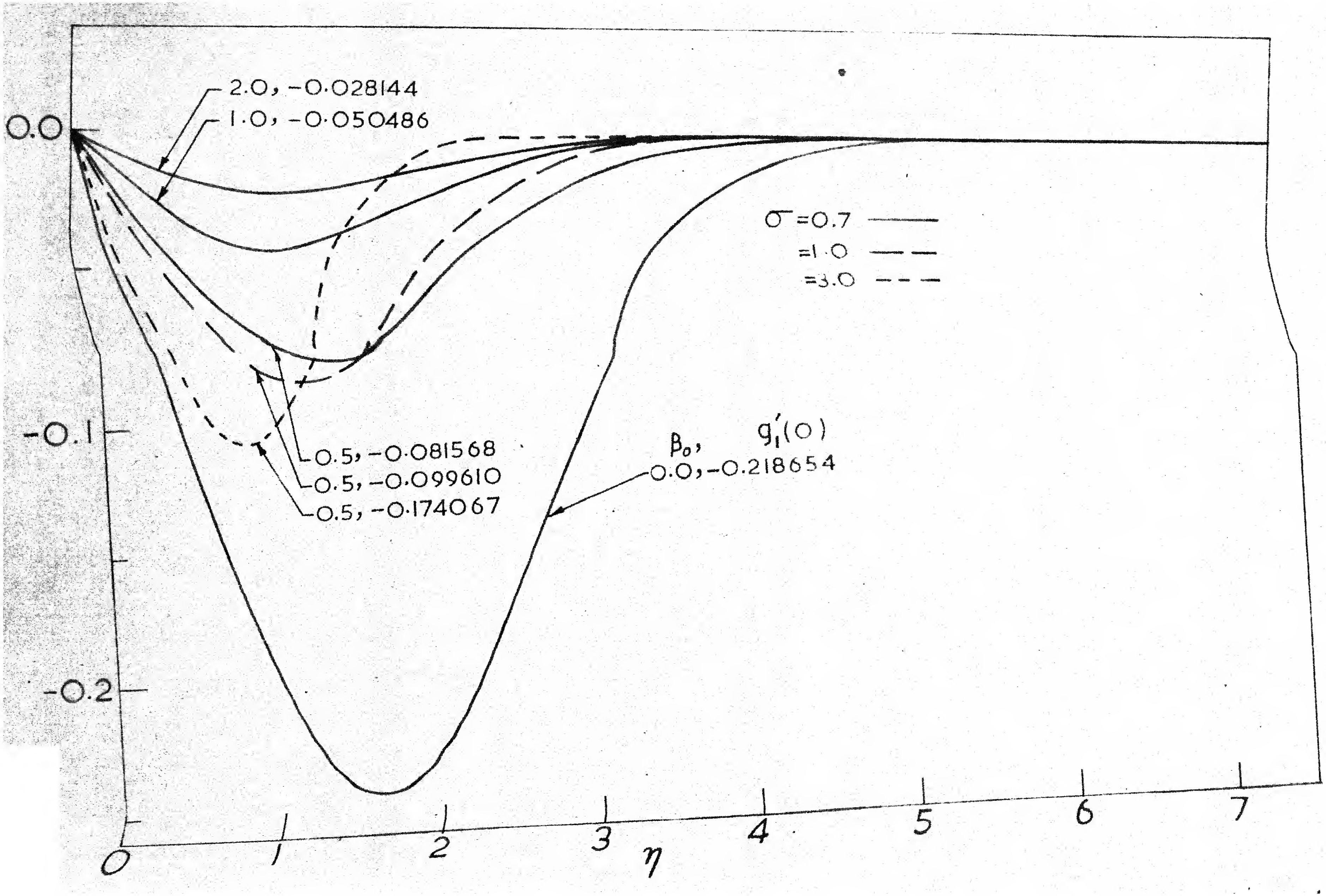


Fig. 5.3_Change in basic velocity profiles in first order boundary layer (Eqn. 5.24a)



2.0, -0.028144
1.0, -0.050486
0.0
-0.1
-0.2
σ =0.7
=1.0
=3.0
0.5, -0.081568
0.5, -0.099610
0.5, -0.174067
β₀, g₁′(0)
0.0, -0.218654
0
1
2
3
4
5
6
7
η

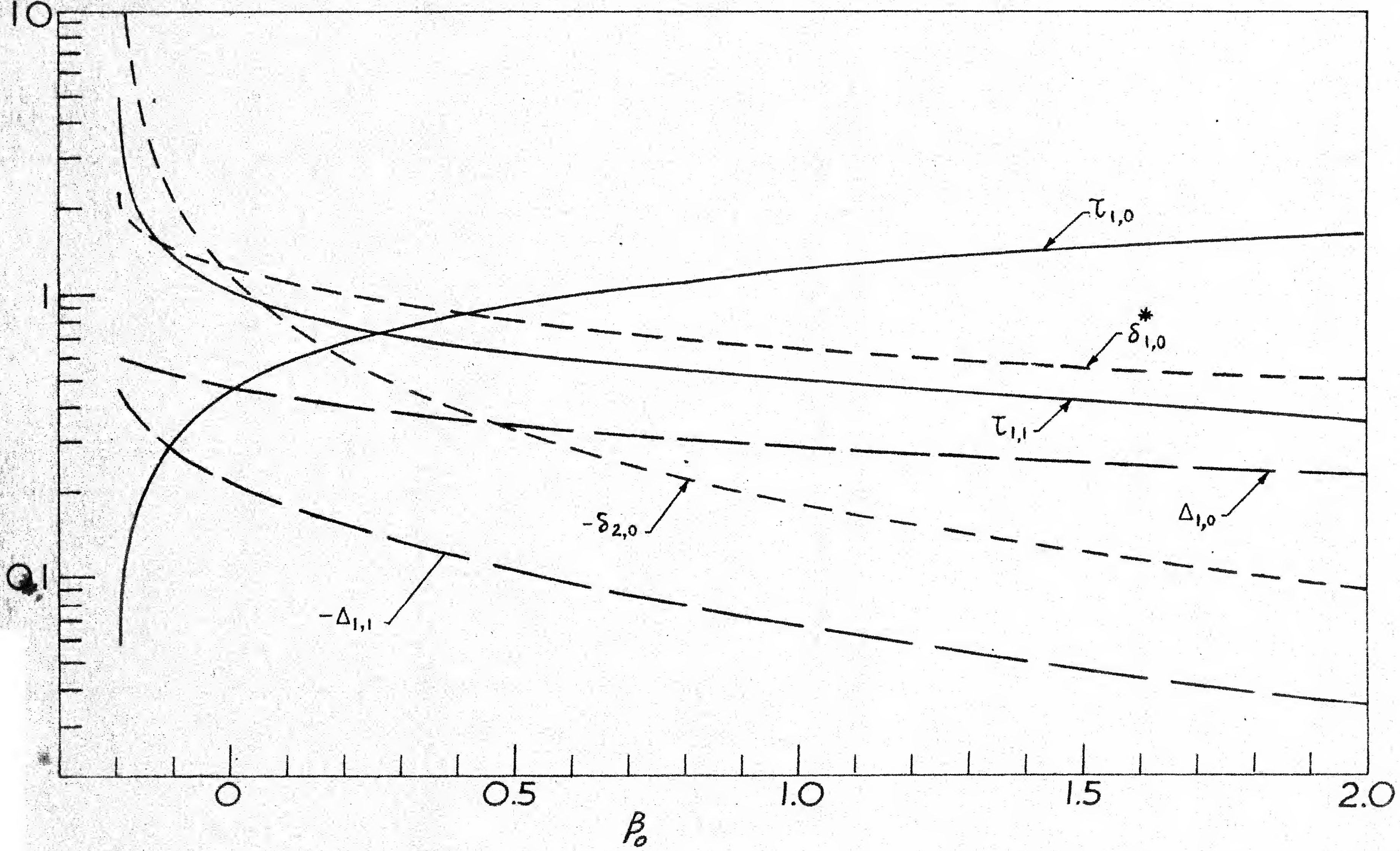


Fig. 5.5 _ Skinfriction, displacement thickness and momentum thickness due to first order boundary layer

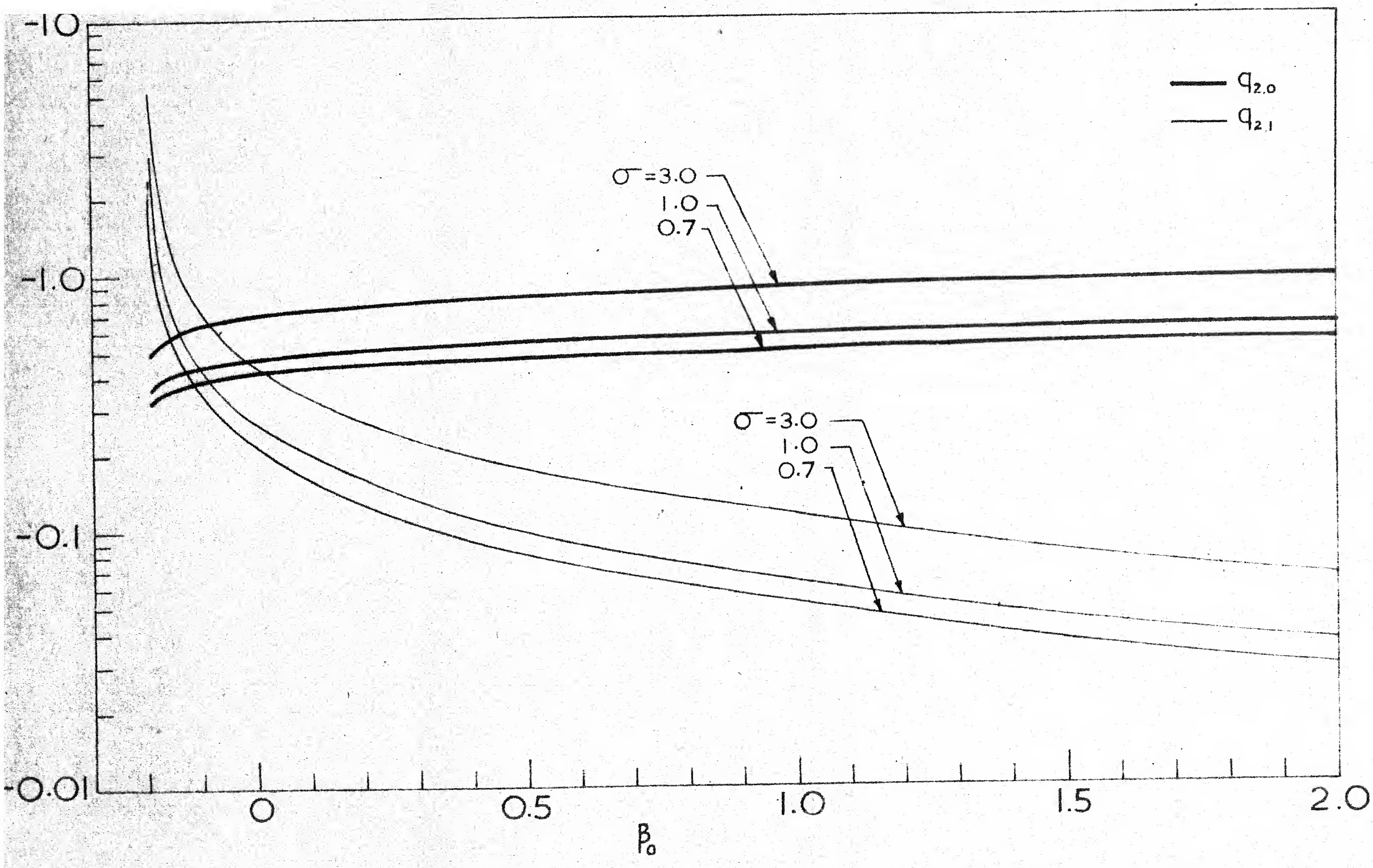


Fig.5.6_Wall heat transfer due to first order boundary layer

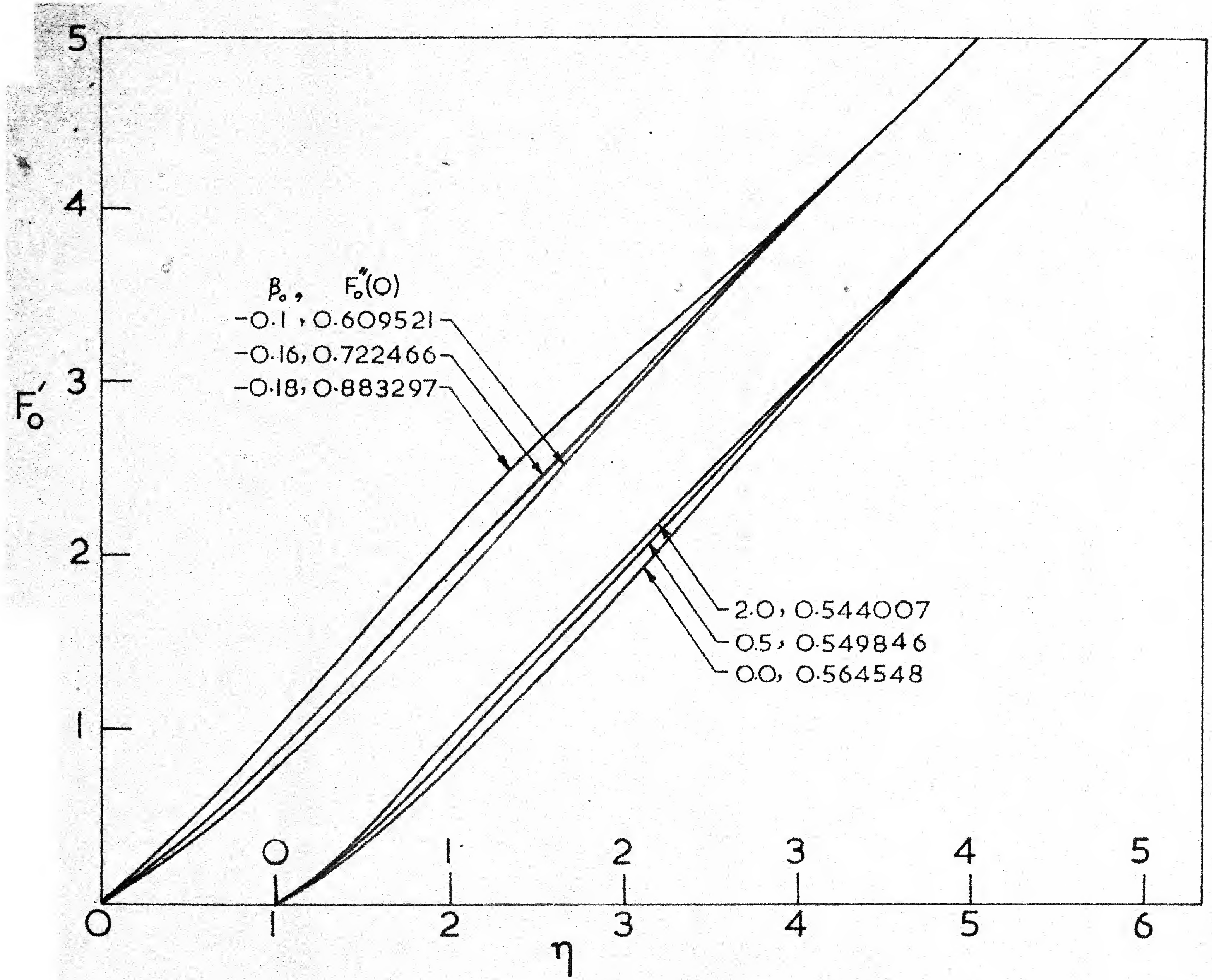


Fig.5.7_ Change in velocity profiles due to transverse curvature (Eqn. 5.55a)

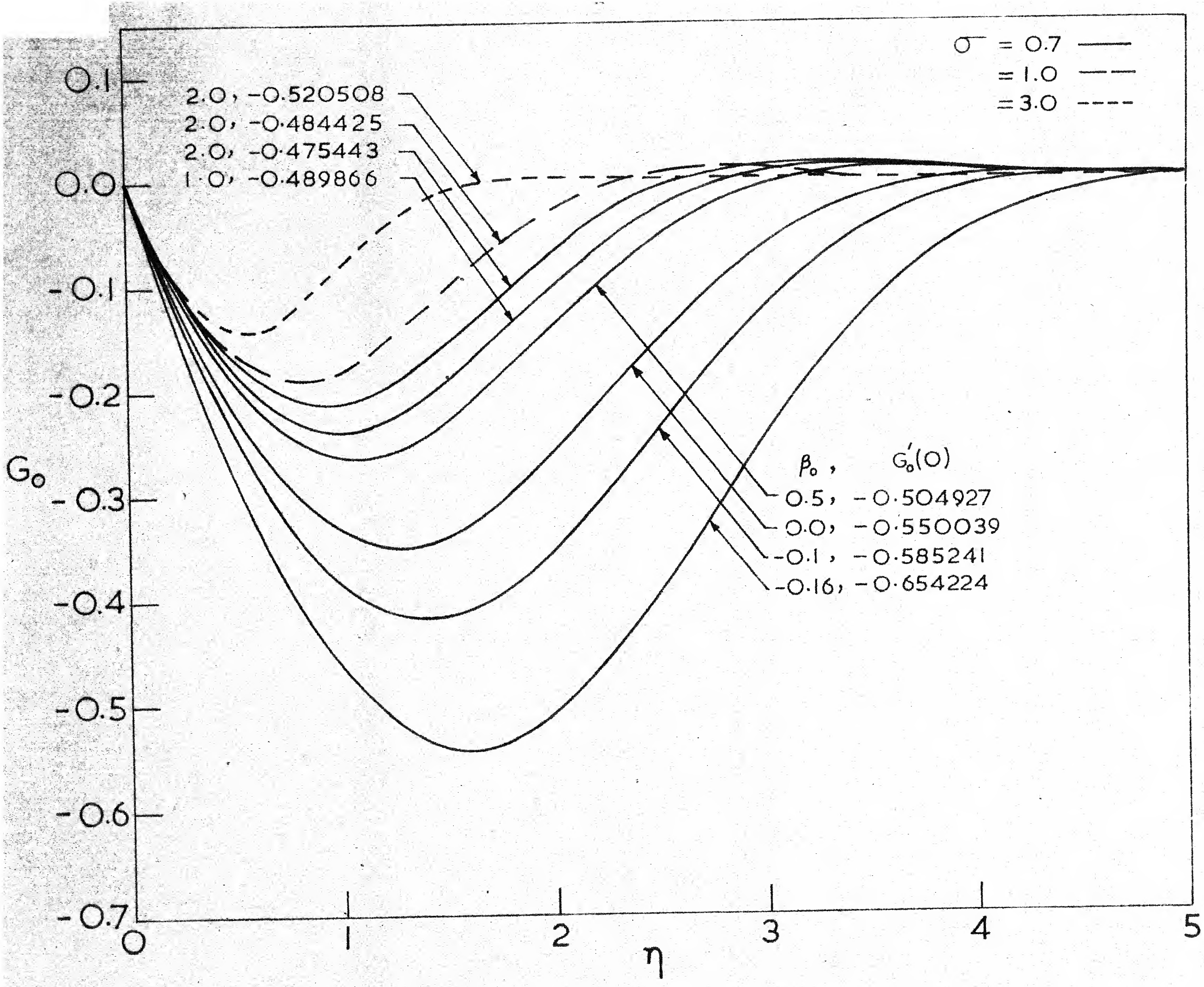


Fig. 5.8_Change in temperature profiles due to transverse

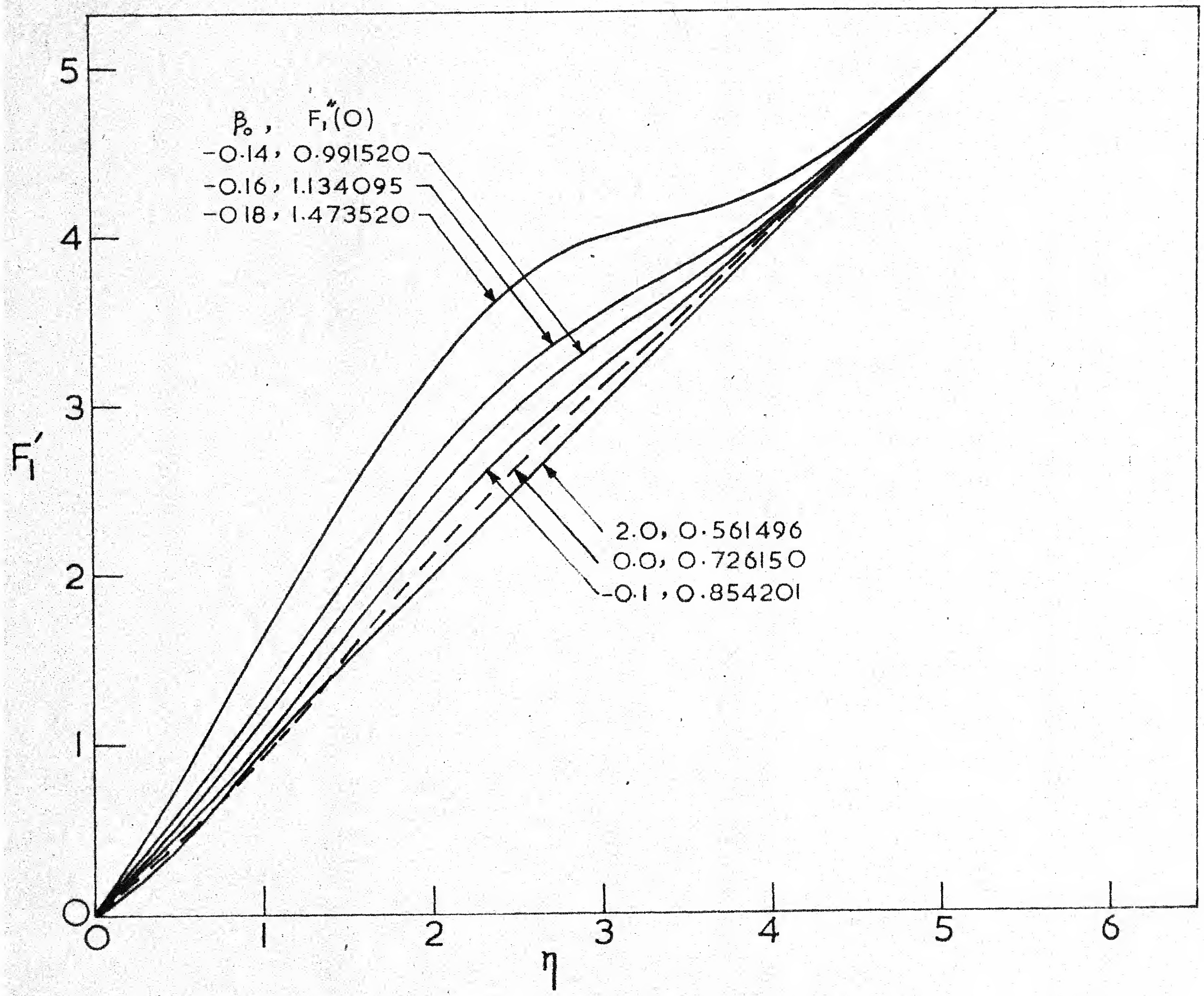


Fig.5.9_Change in velocity profiles due to transverse curvature (Eqn. 5.56a)

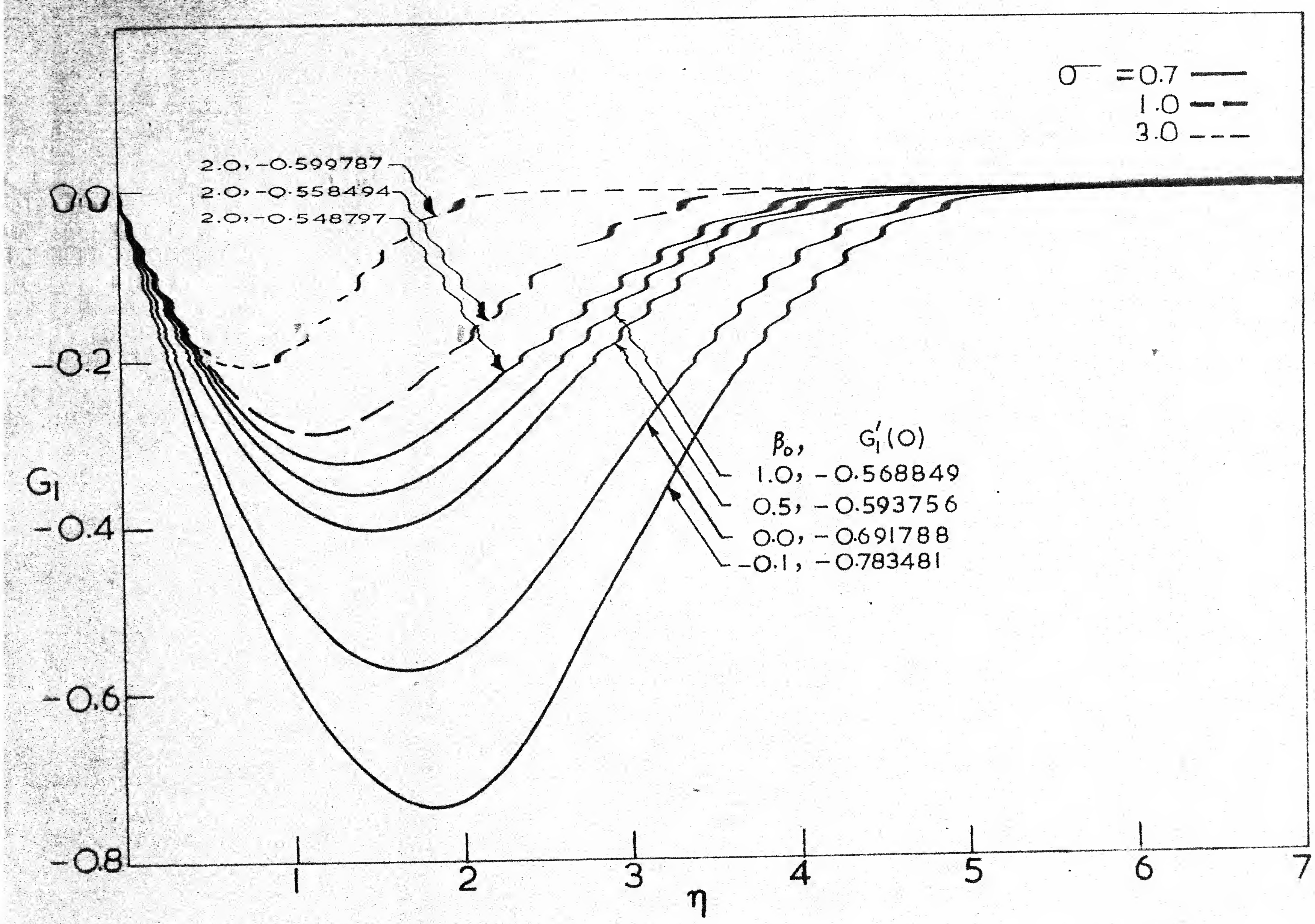


Fig.5.10_Change in temperature profiles due to transverse curvature (Eqn.5.56b)

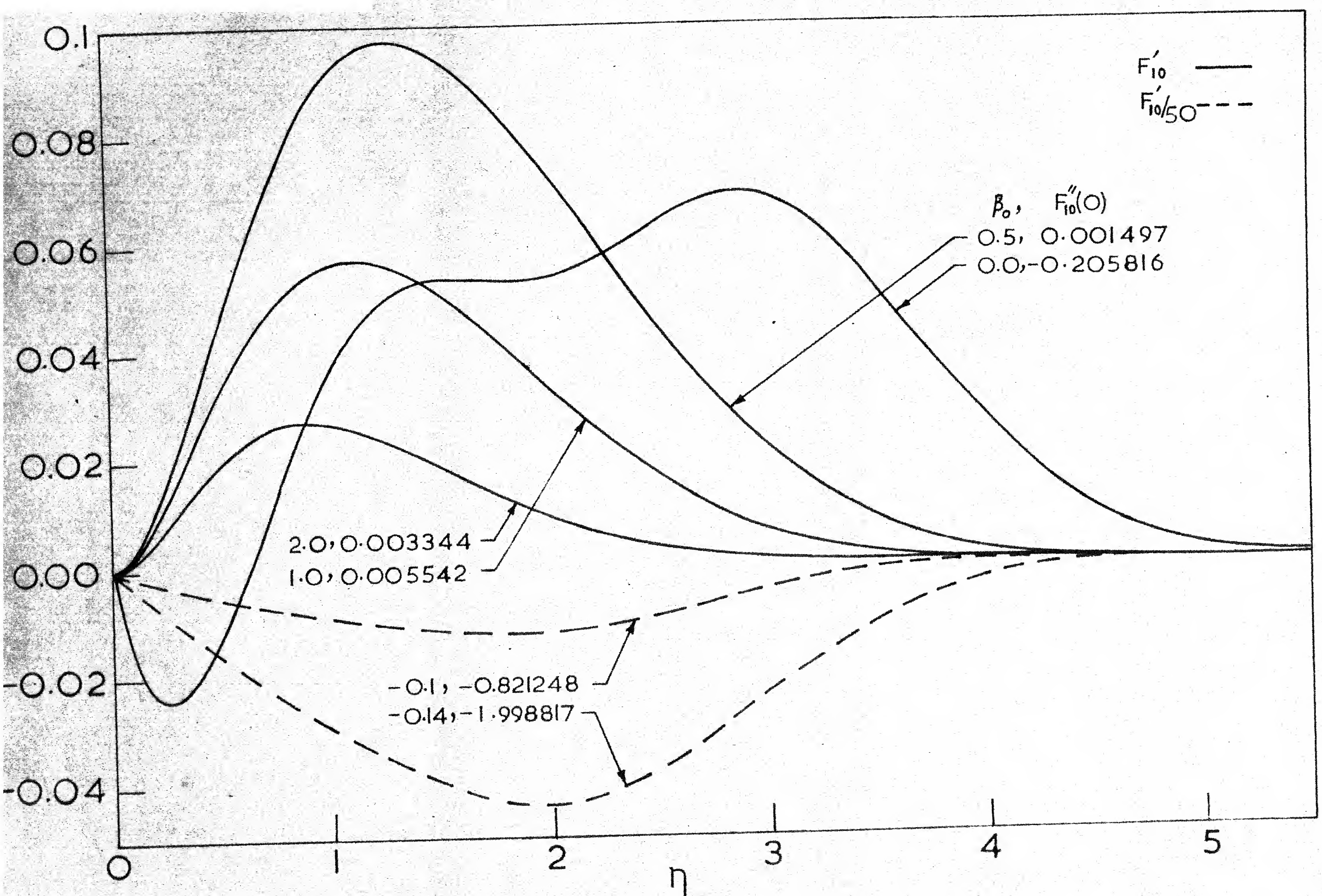


Fig. 5.11_Change in velocity profiles due to transverse curvature (Eqn. 5.57a)

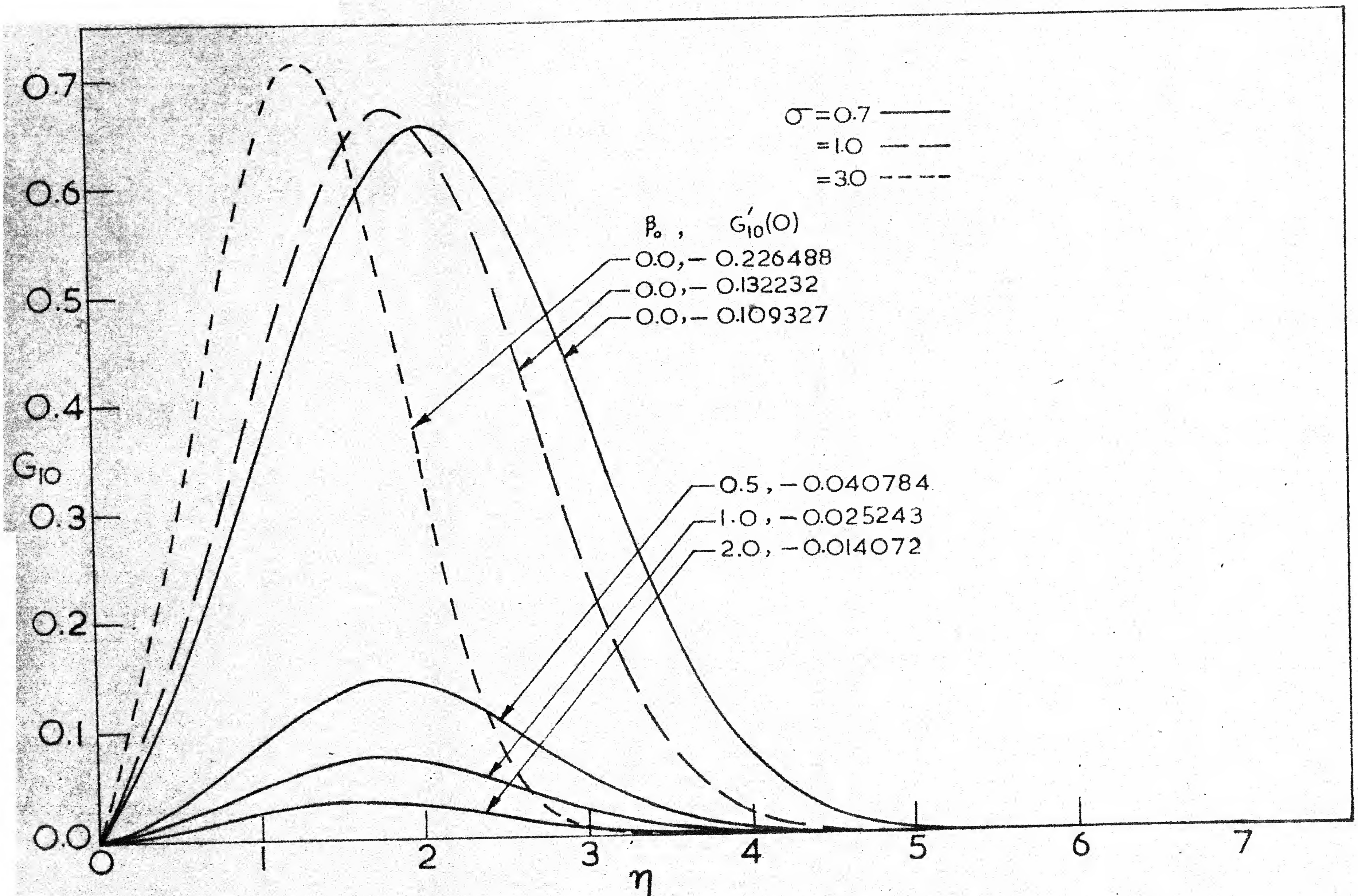


Fig.5.12 _ Change in temperature profiles due to transverse curvature (Eqn. 5.57b)

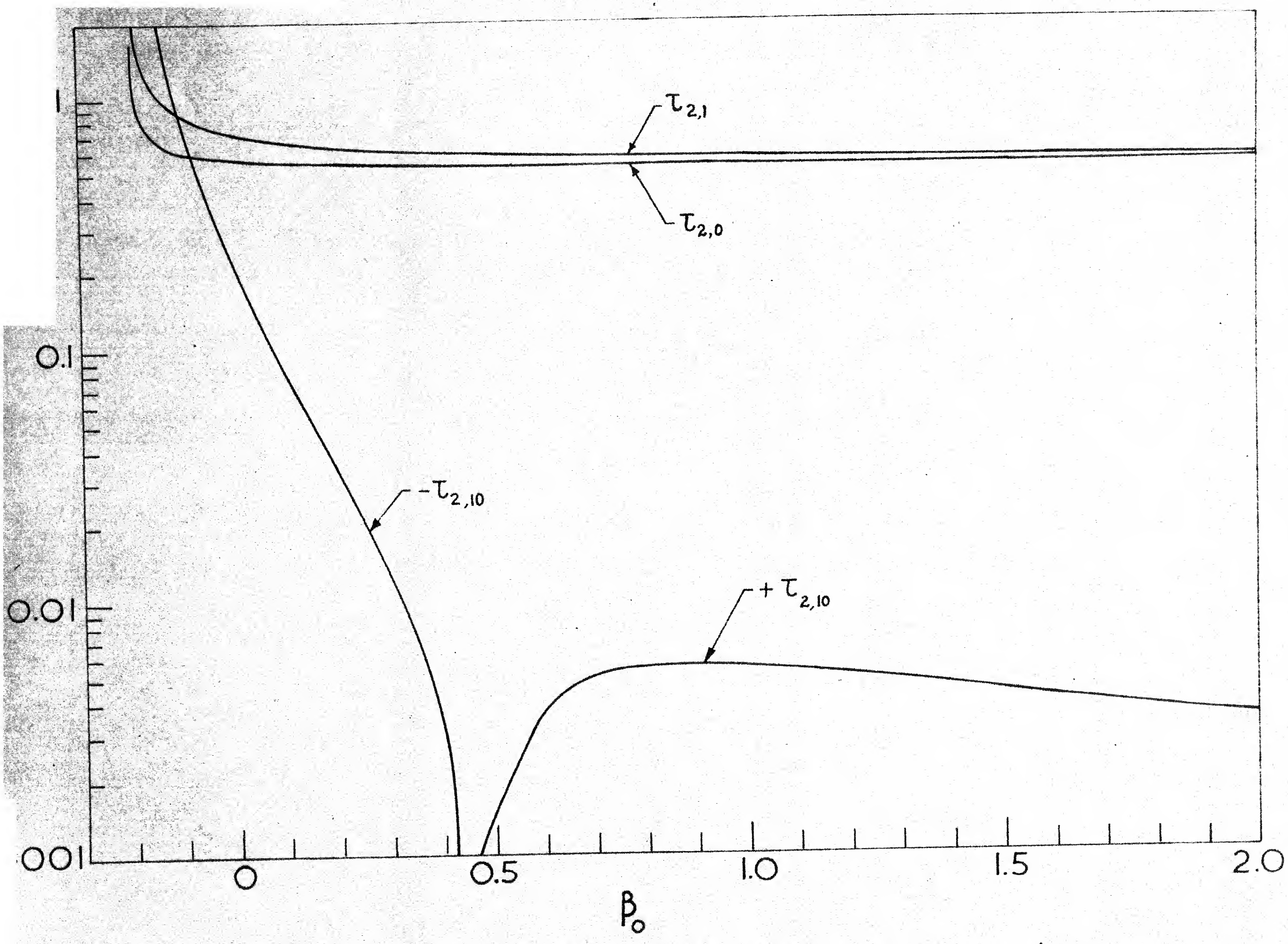


Fig 5.13– Change in skin friction due to transverse curvature.

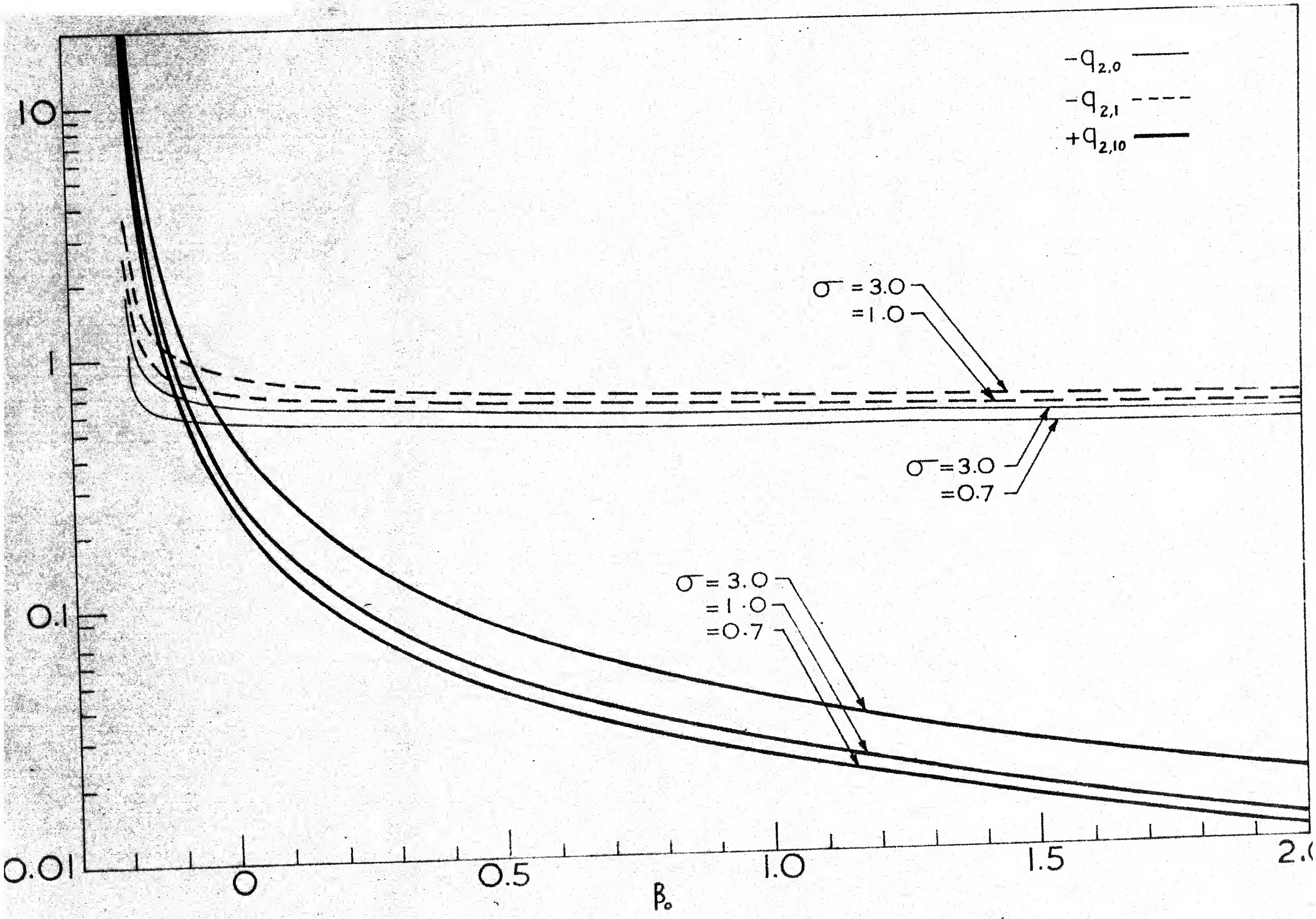


Fig. 5.14_Change in wall heat transfer due to transverse curvature

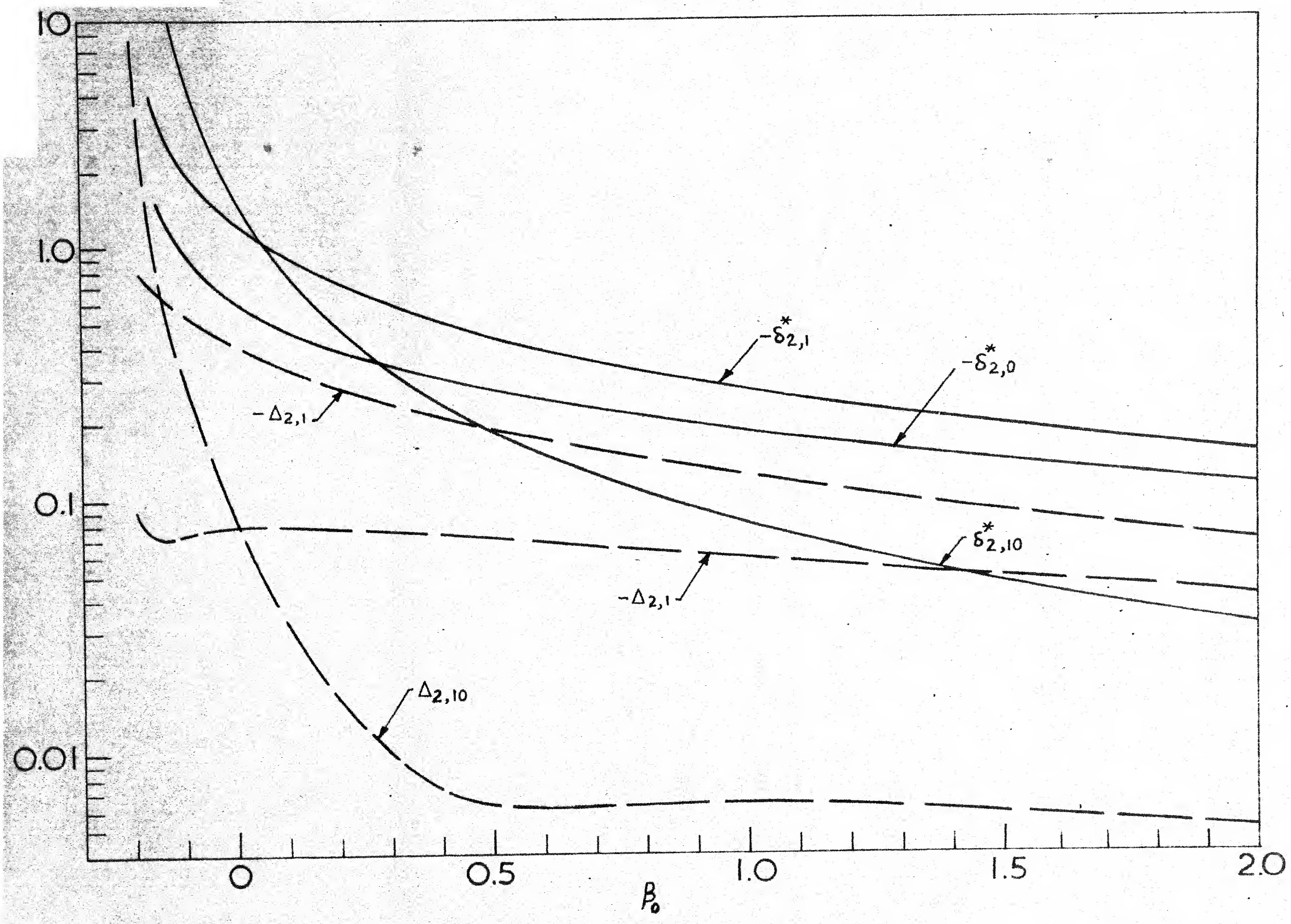


Fig. 5.15_Change in displacement and momentum thicknesses due to

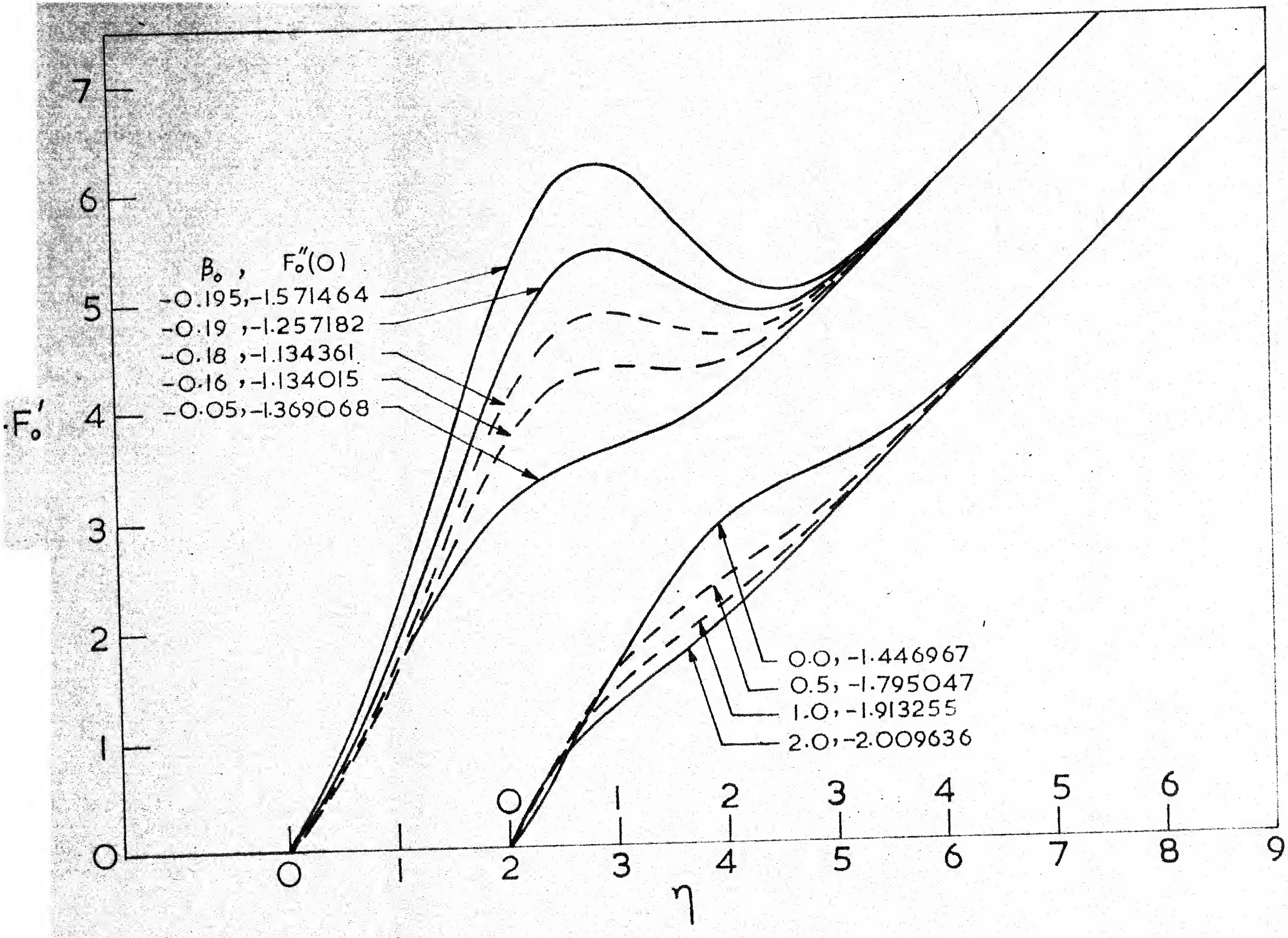


Fig.5.16_Change in velocity profiles due to longitudinal curvature (Eqn. 5.55a)

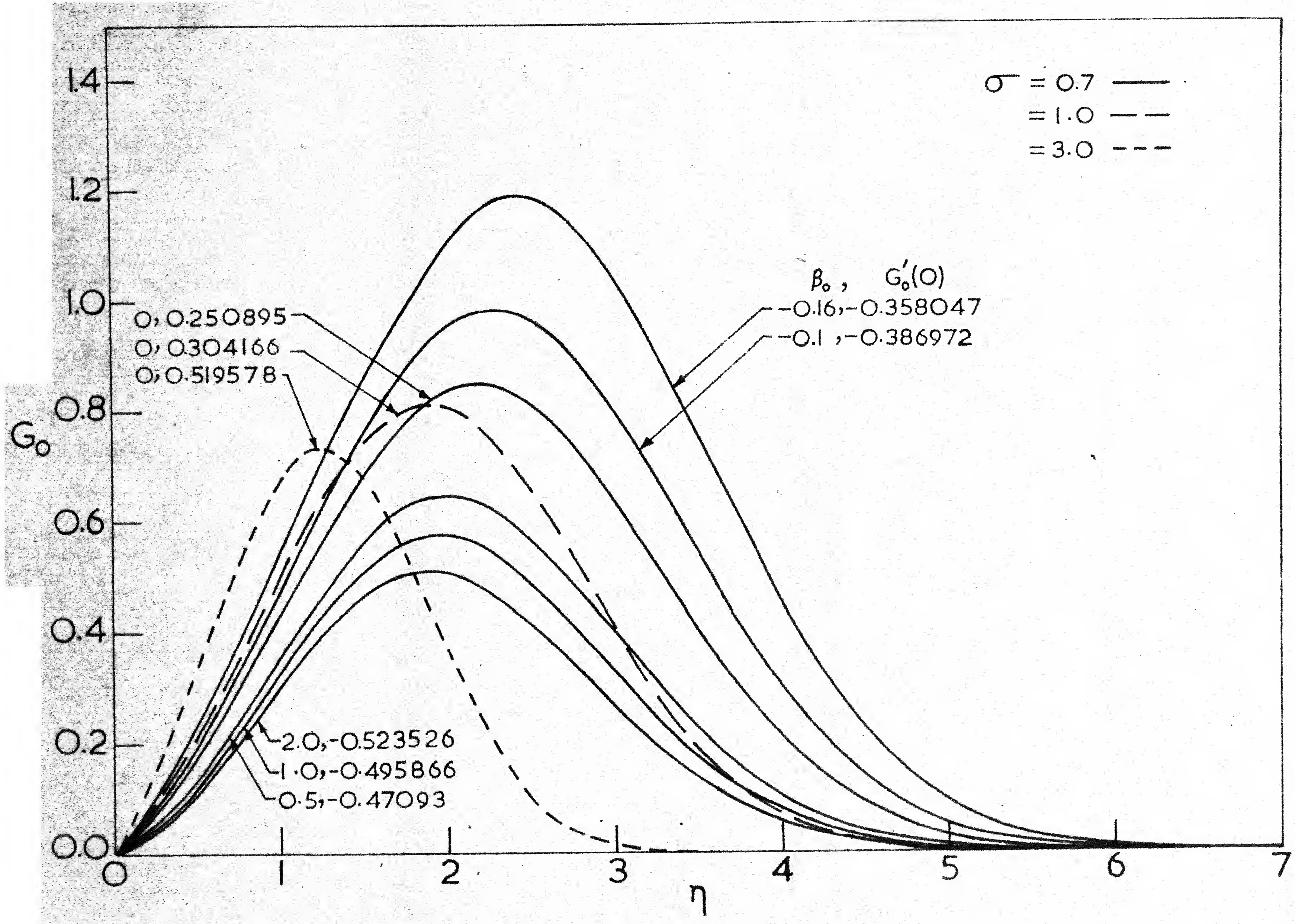


Fig. 5.17_Change in temperature profiles due to longitudinal curvature (Eqn. 5.76 b)

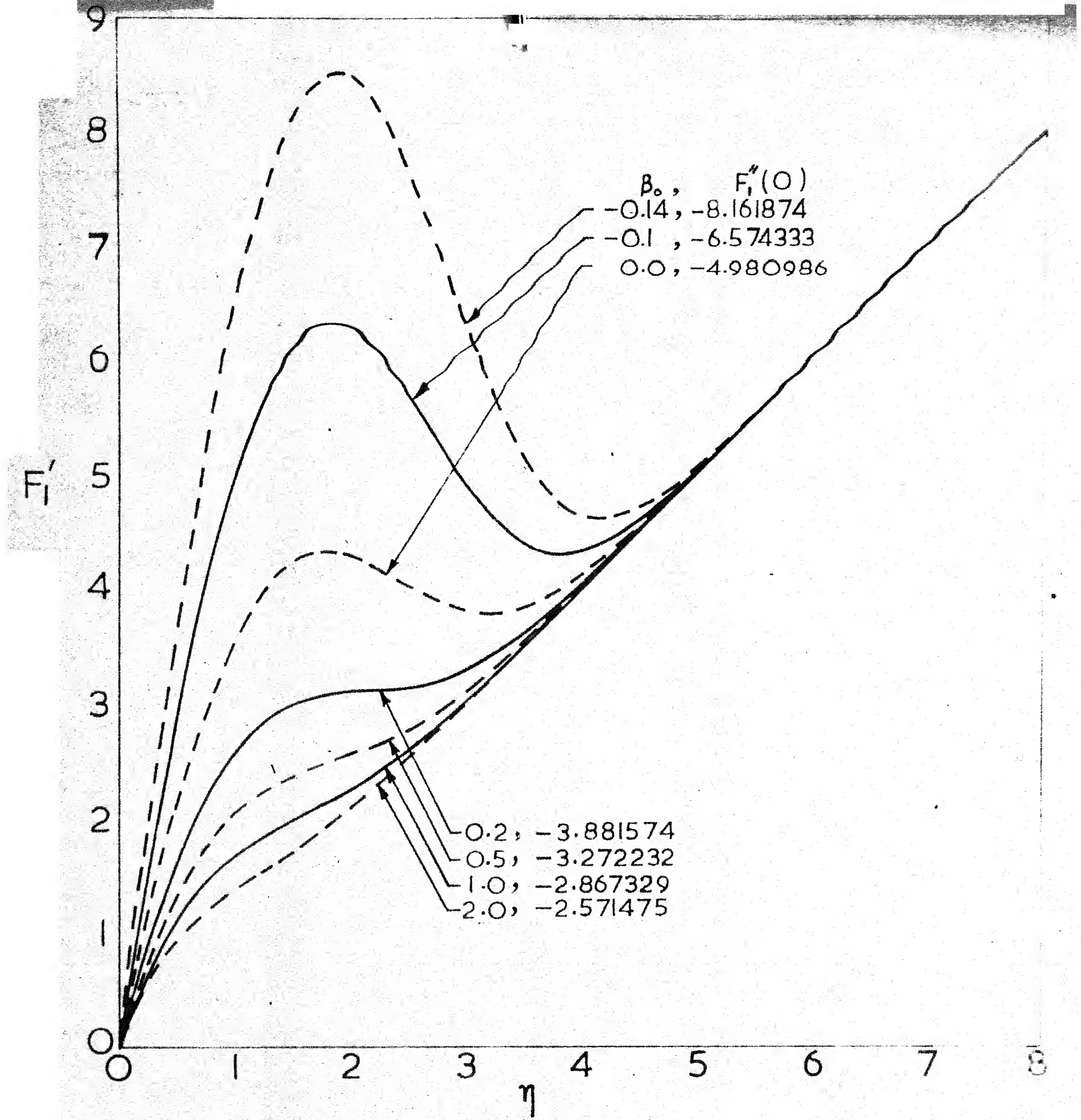


Fig.5.18_ Change in velocity profile due to longitudinal curvature (Eqn. 5.77a)

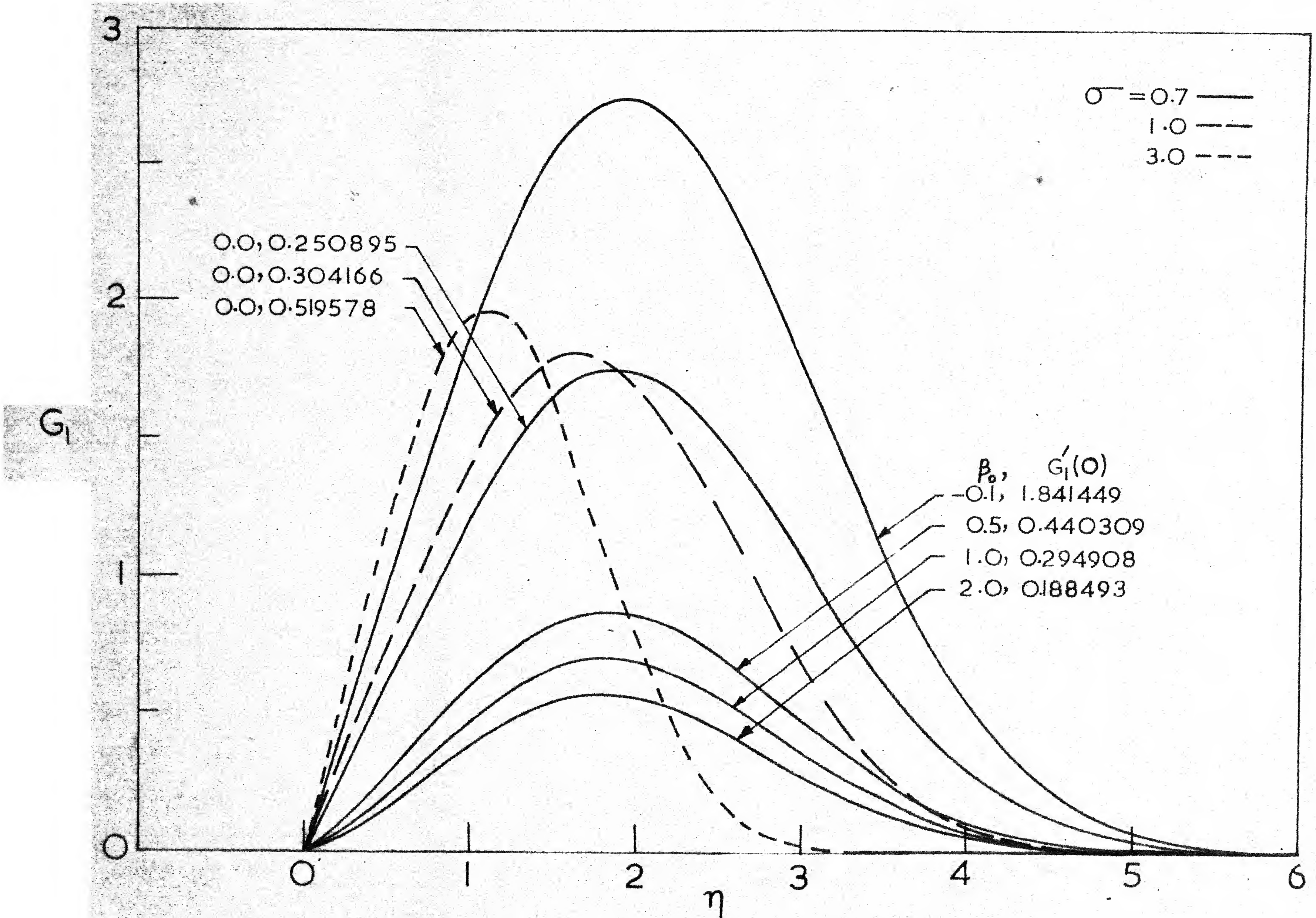


Fig. 5.19_Change in temperature profiles due to longitudinal curvature (Eqn. 5.77b)

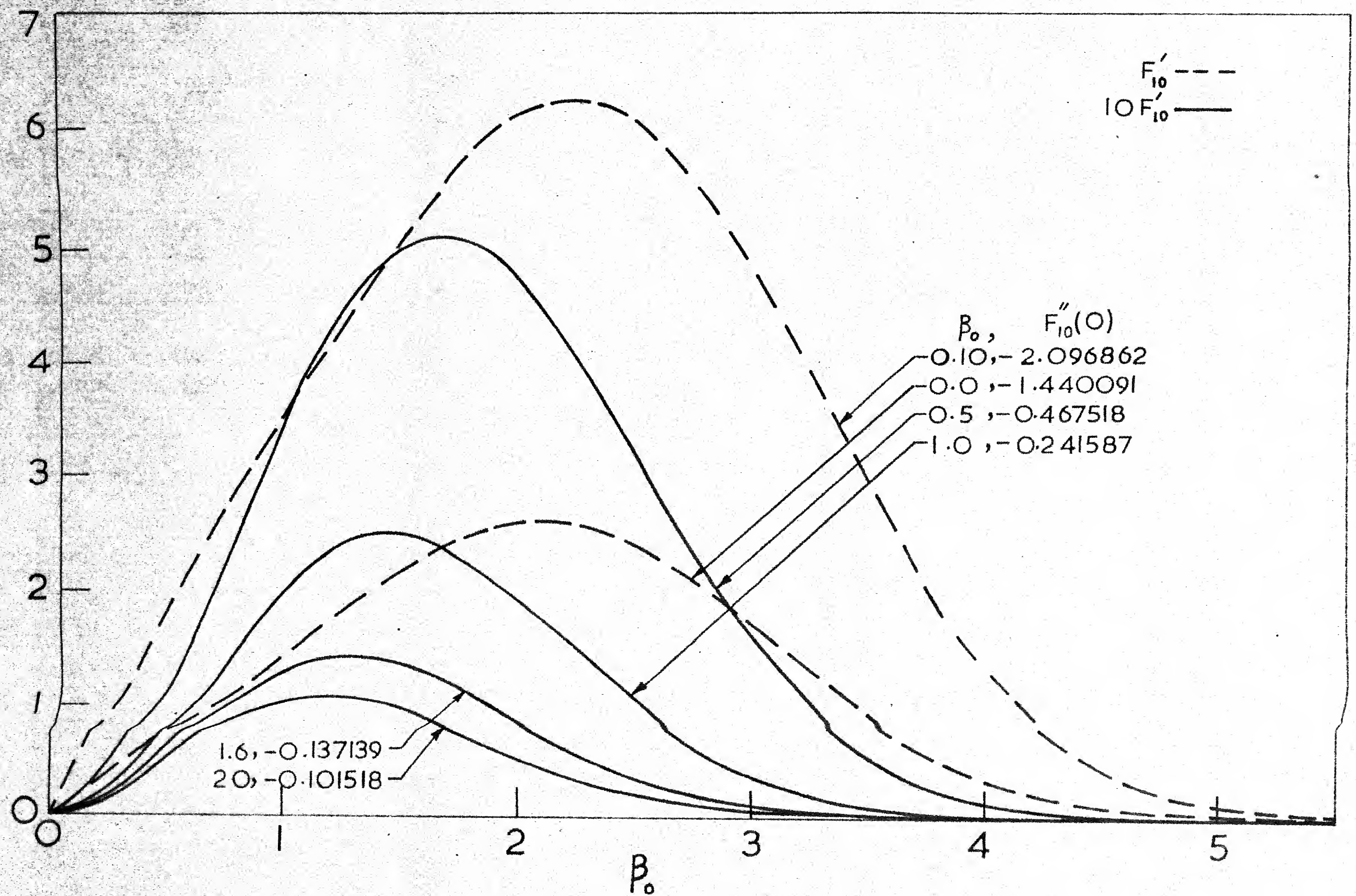


Fig. 5.20 Change in velocity profiles due to longitudinal curvature (Eqn. 5.78a)

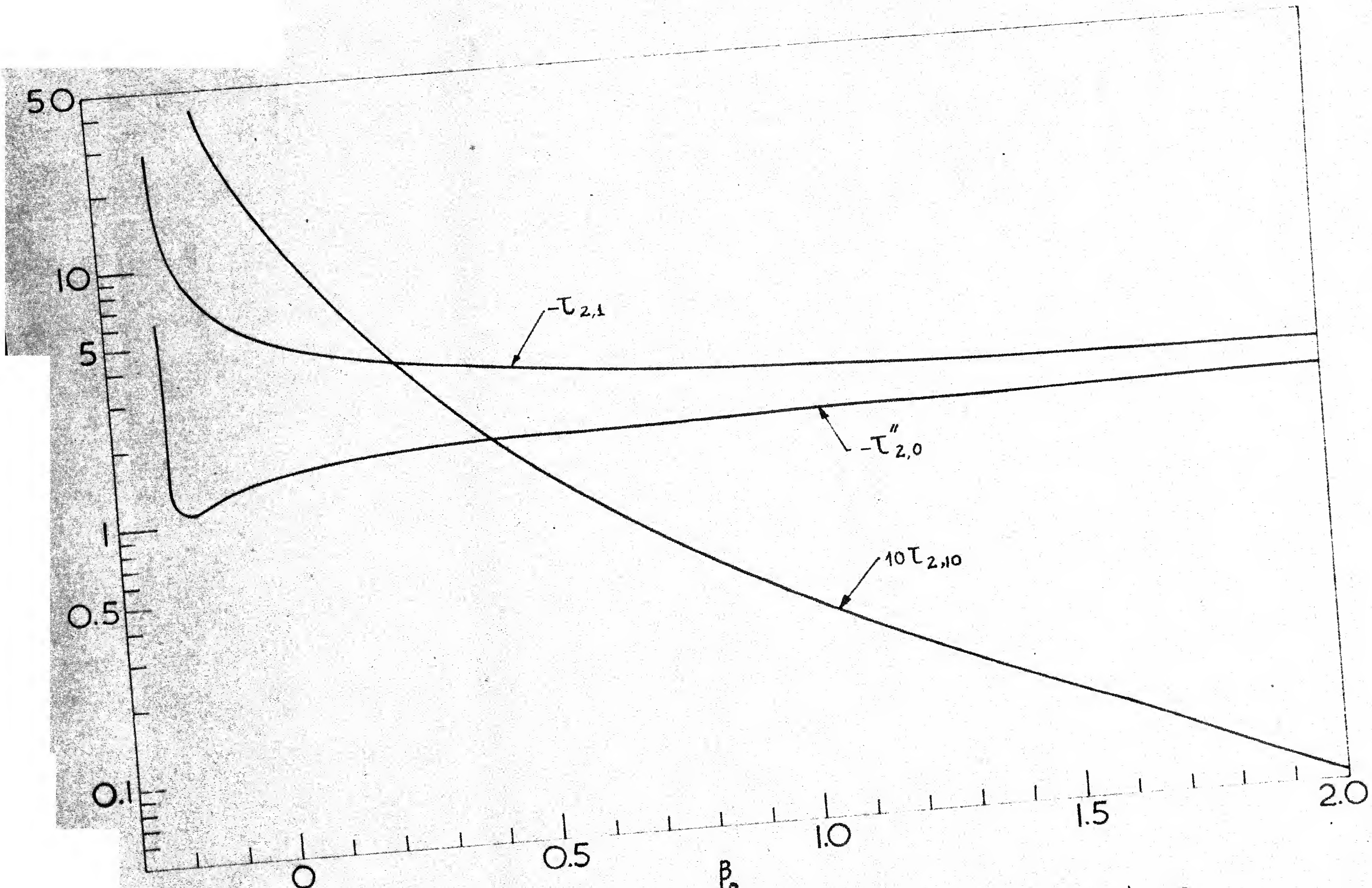


Fig.5.22_ Change in skin friction due to longitudinal curvature

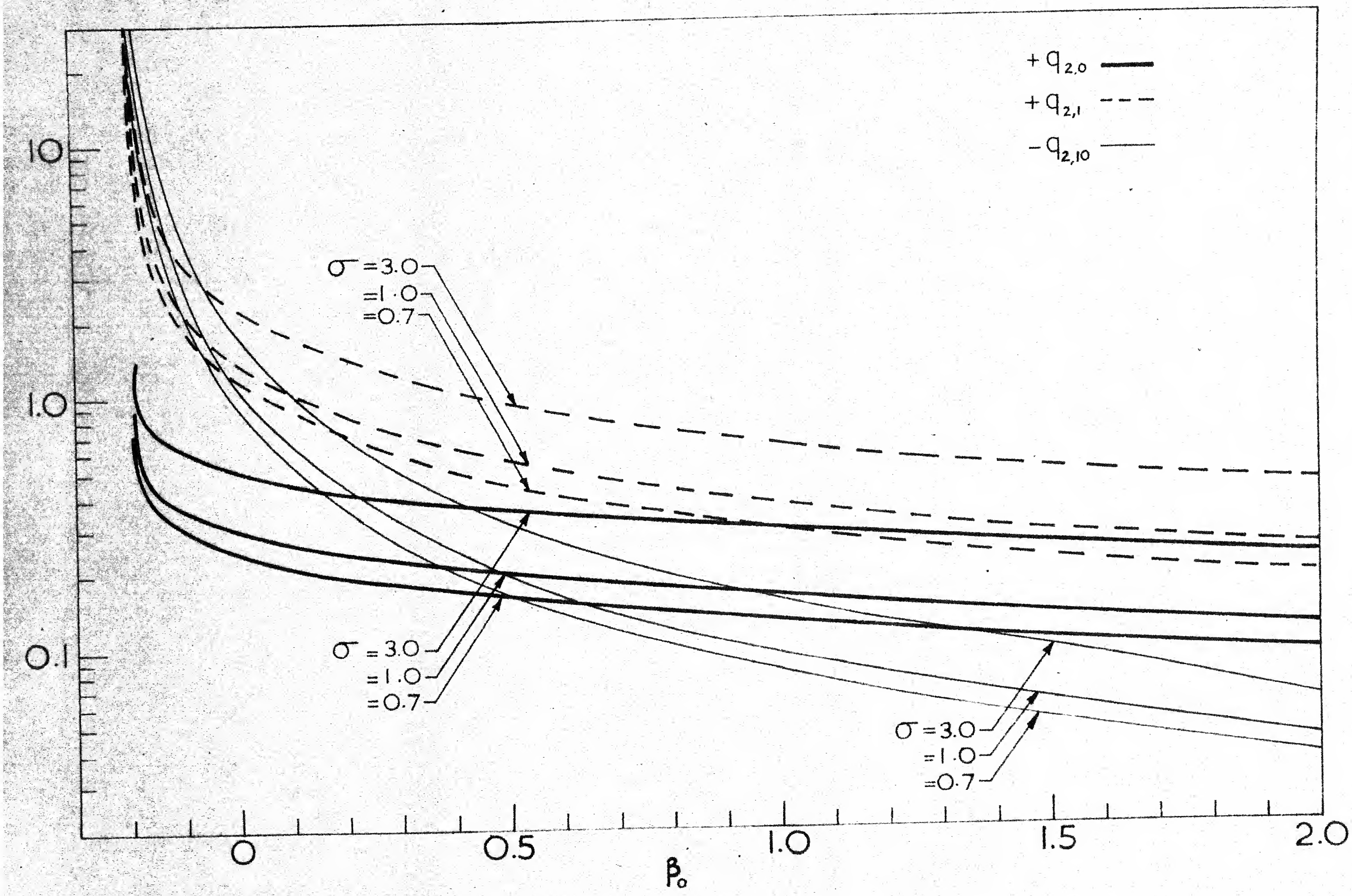


Fig.5.23_Change in wall heat transfer due to longitudinal curvature

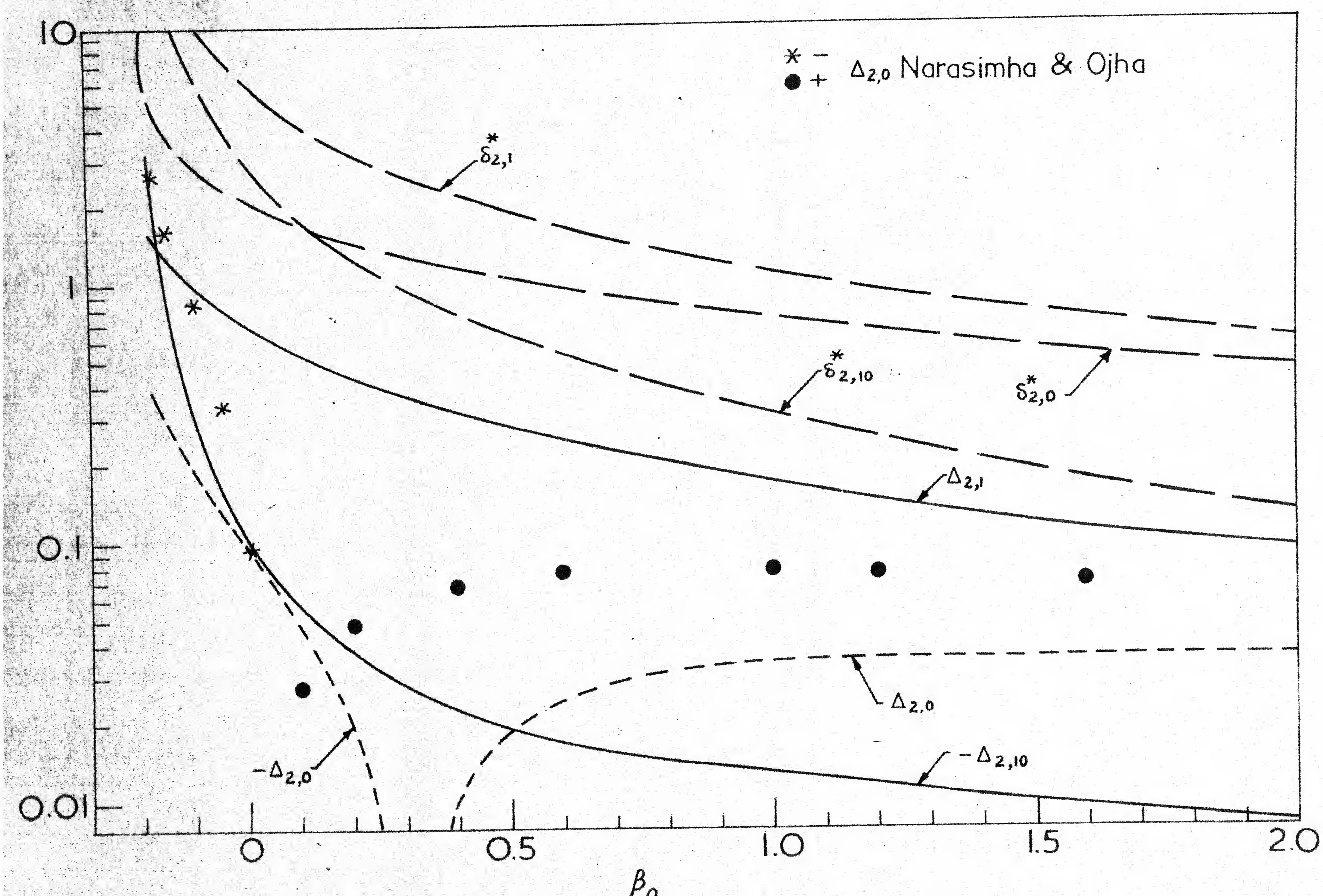


Fig.5.24_Change in displacement and momentum thicknesses due to longitudinal curvature

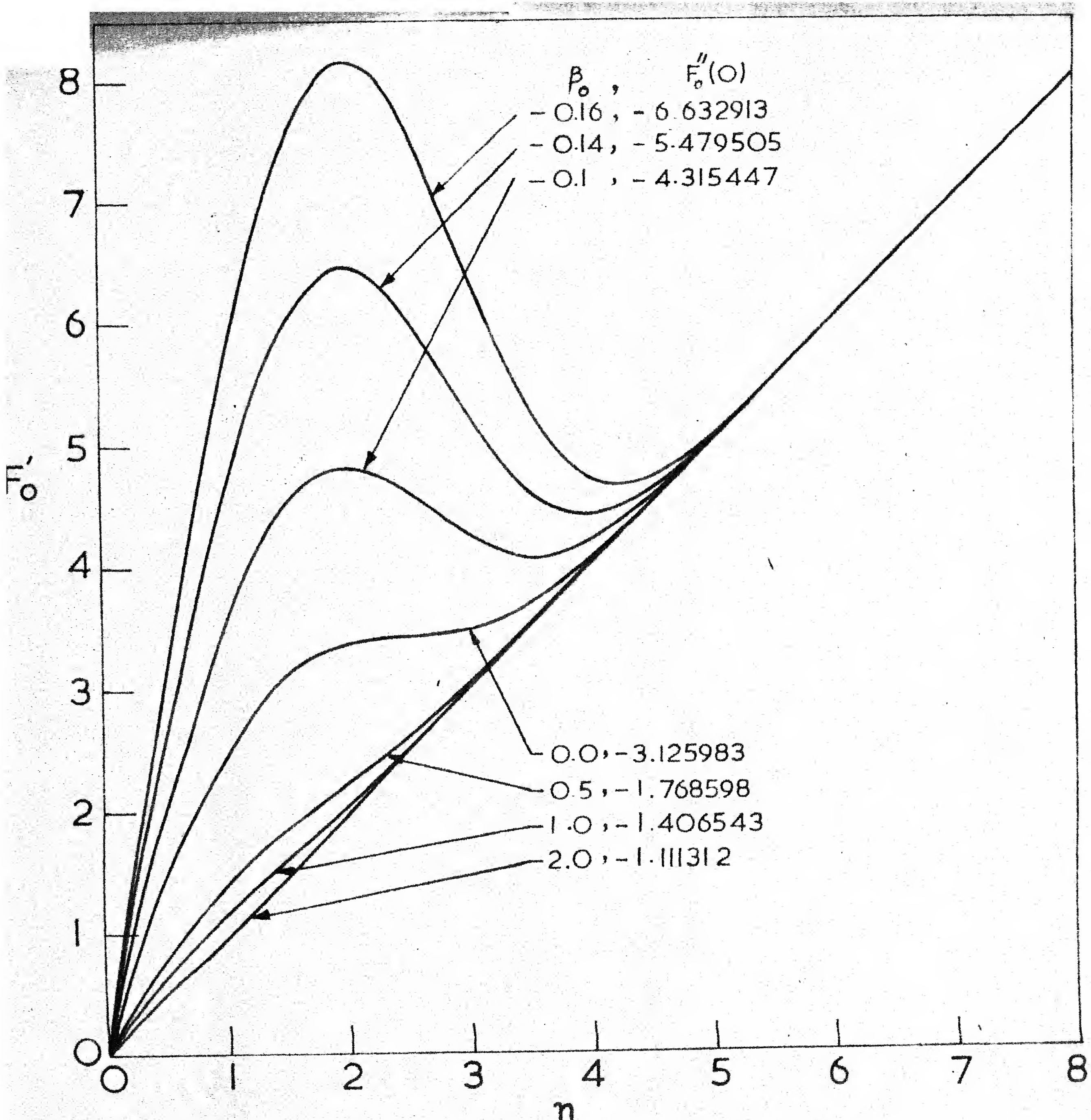


Fig. 5.25 – Change in velocity profile due to external vorticity (Eqn. 5.97a)

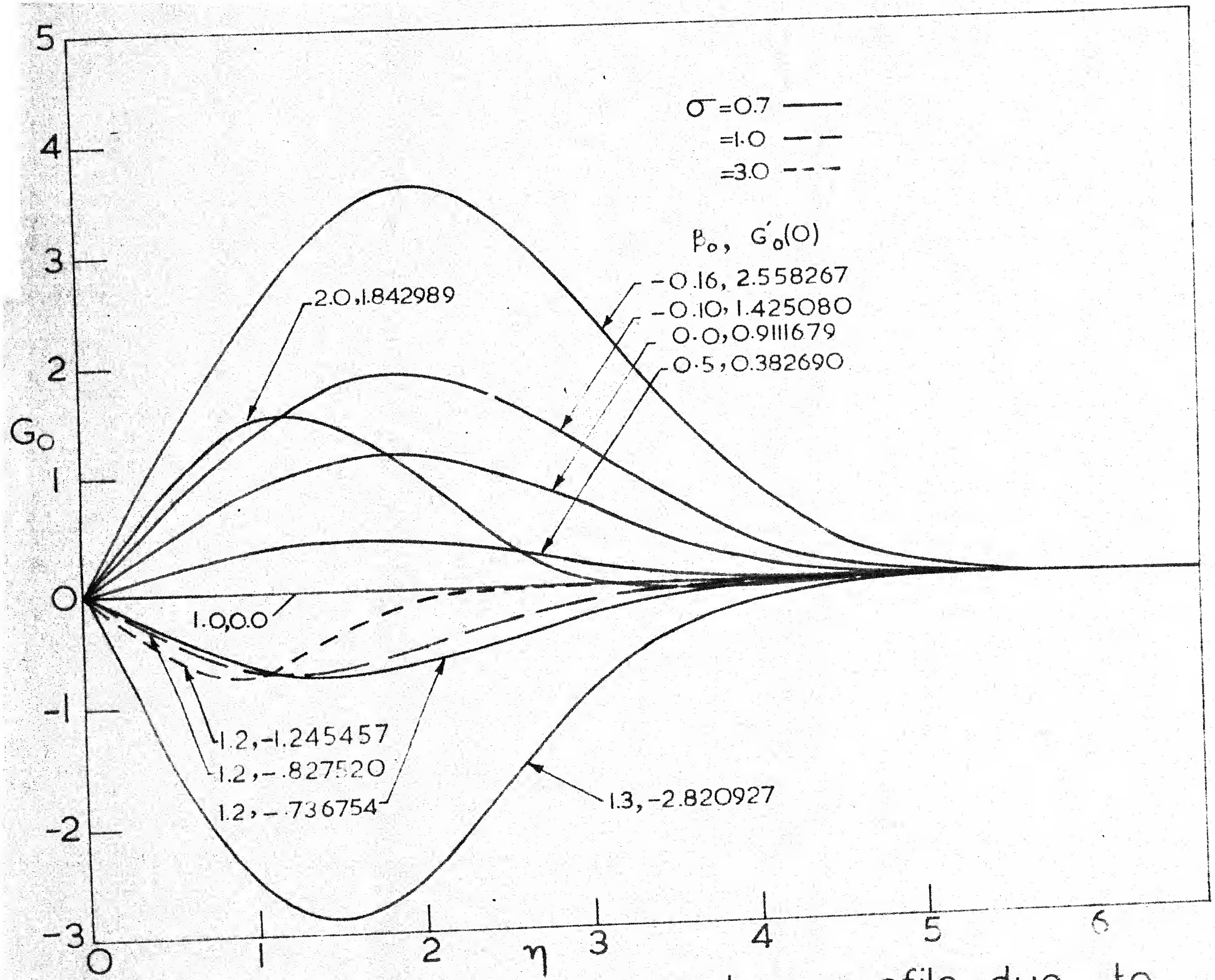


Fig.5.26_Change in temperature profile due to external vorticity (Eqn.5.97b)

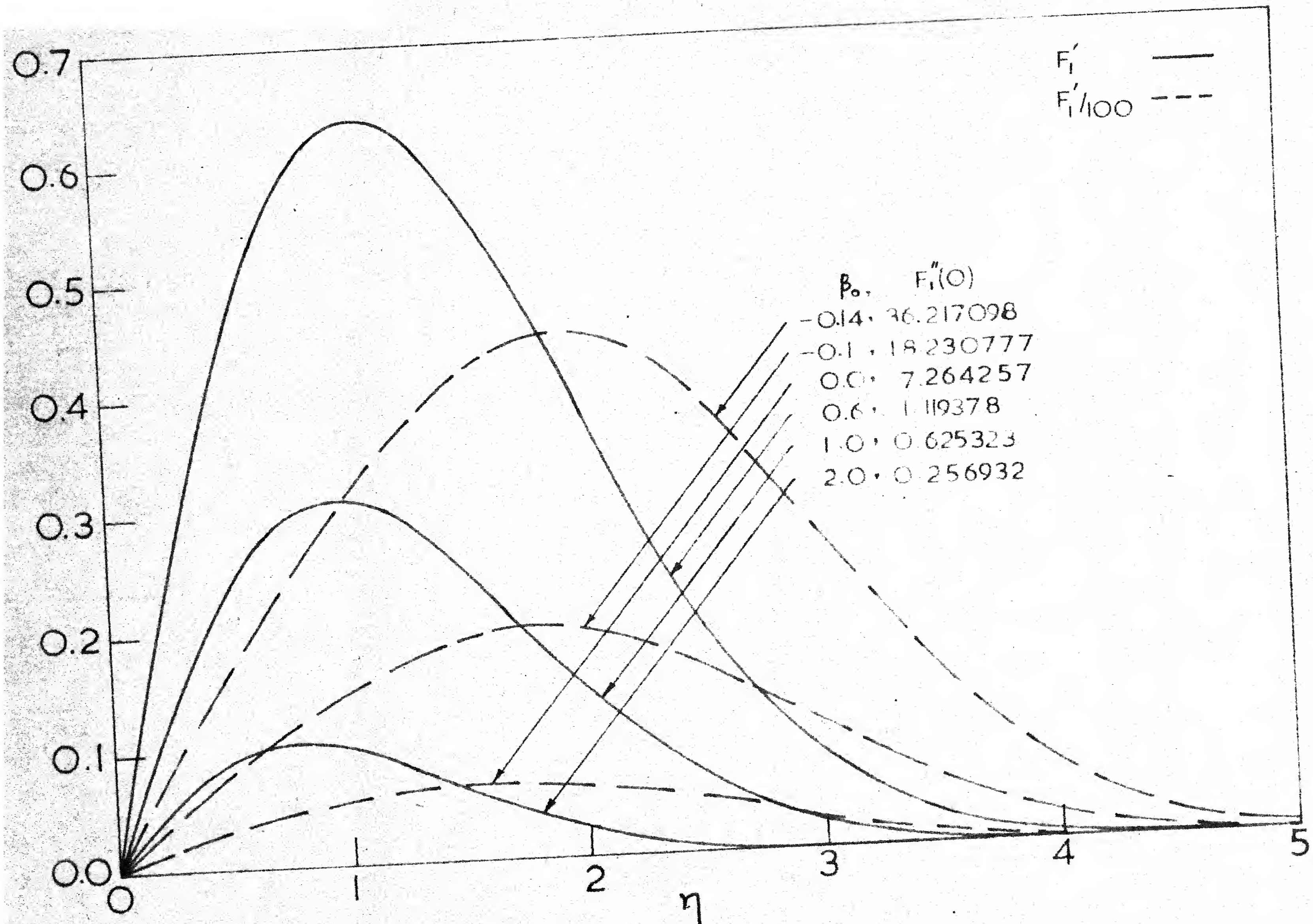
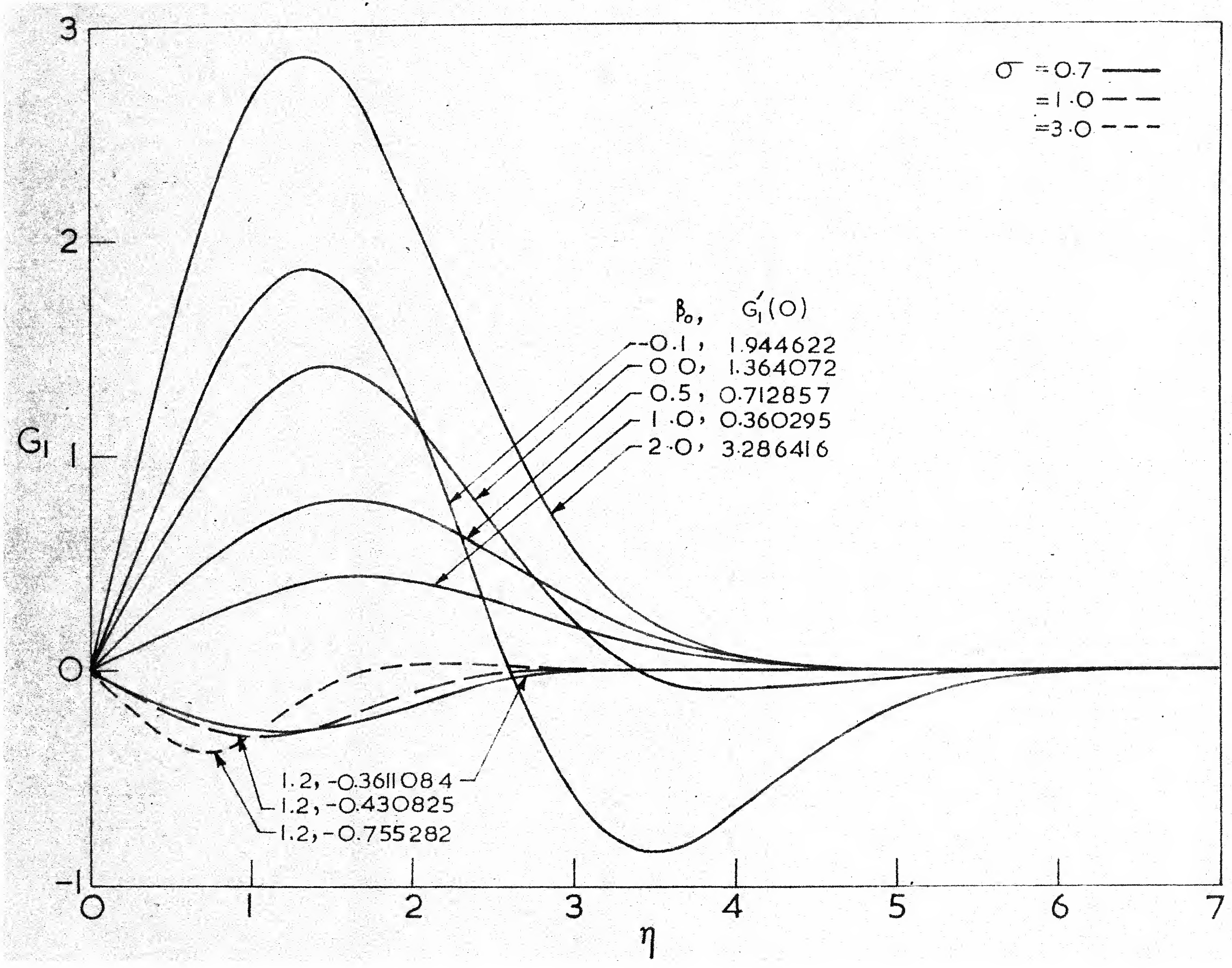


Fig.5.27 _ Change in velocity profiles due to external vorticity (Eqn.5.98a)



σ = 0.7
= 1.0
= 3.0
β_o, $G_1'(0)$
-0.1, 1.944622
0.0, 1.364072
0.5, 0.712857
1.0, 0.360295
2.0, 3.286416
1.2, -0.3611084
1.2, -0.430825
1.2, -0.755282
G_1
η

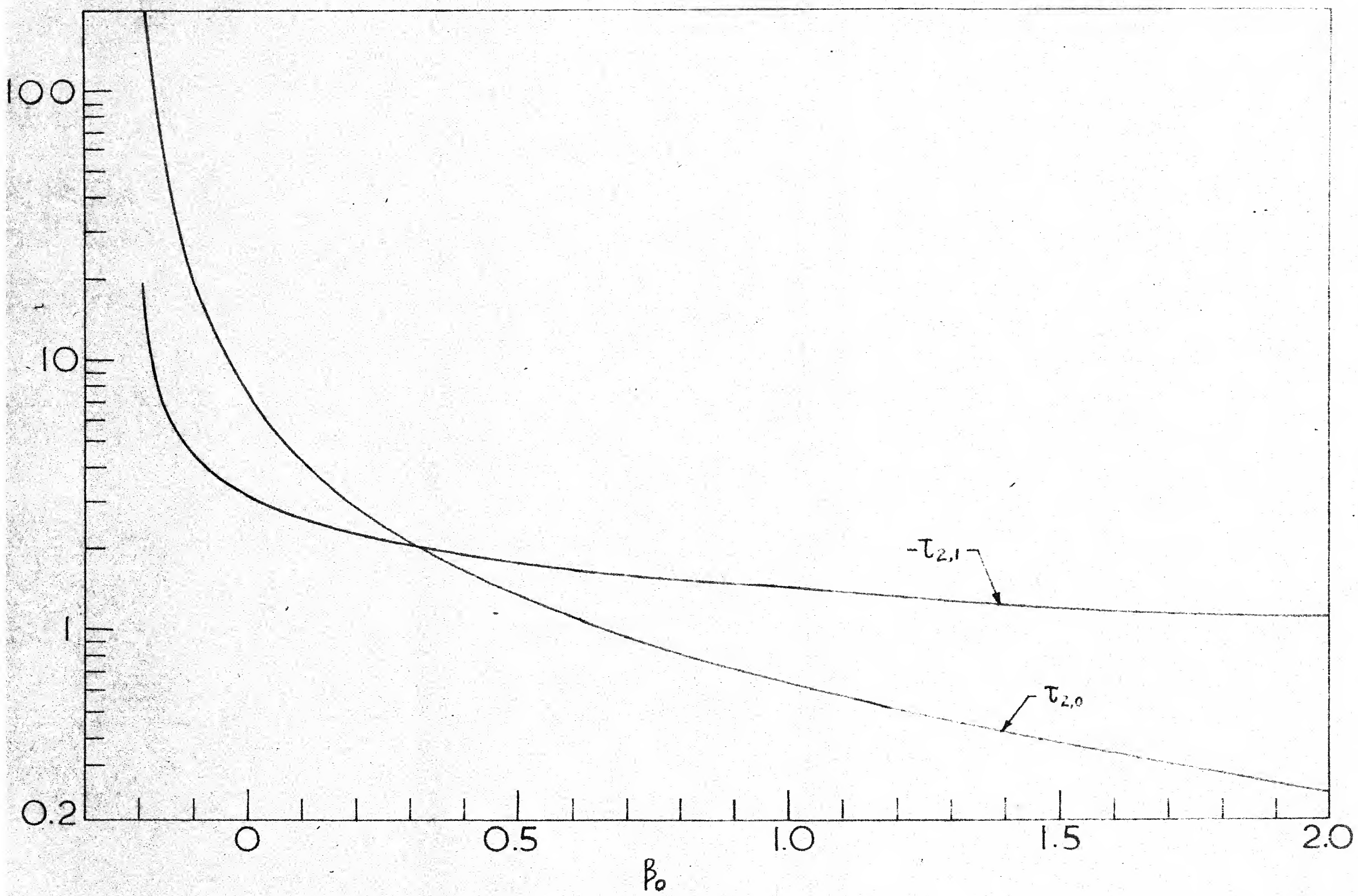


Fig.5.29_Change in skin friction due to external vorticity

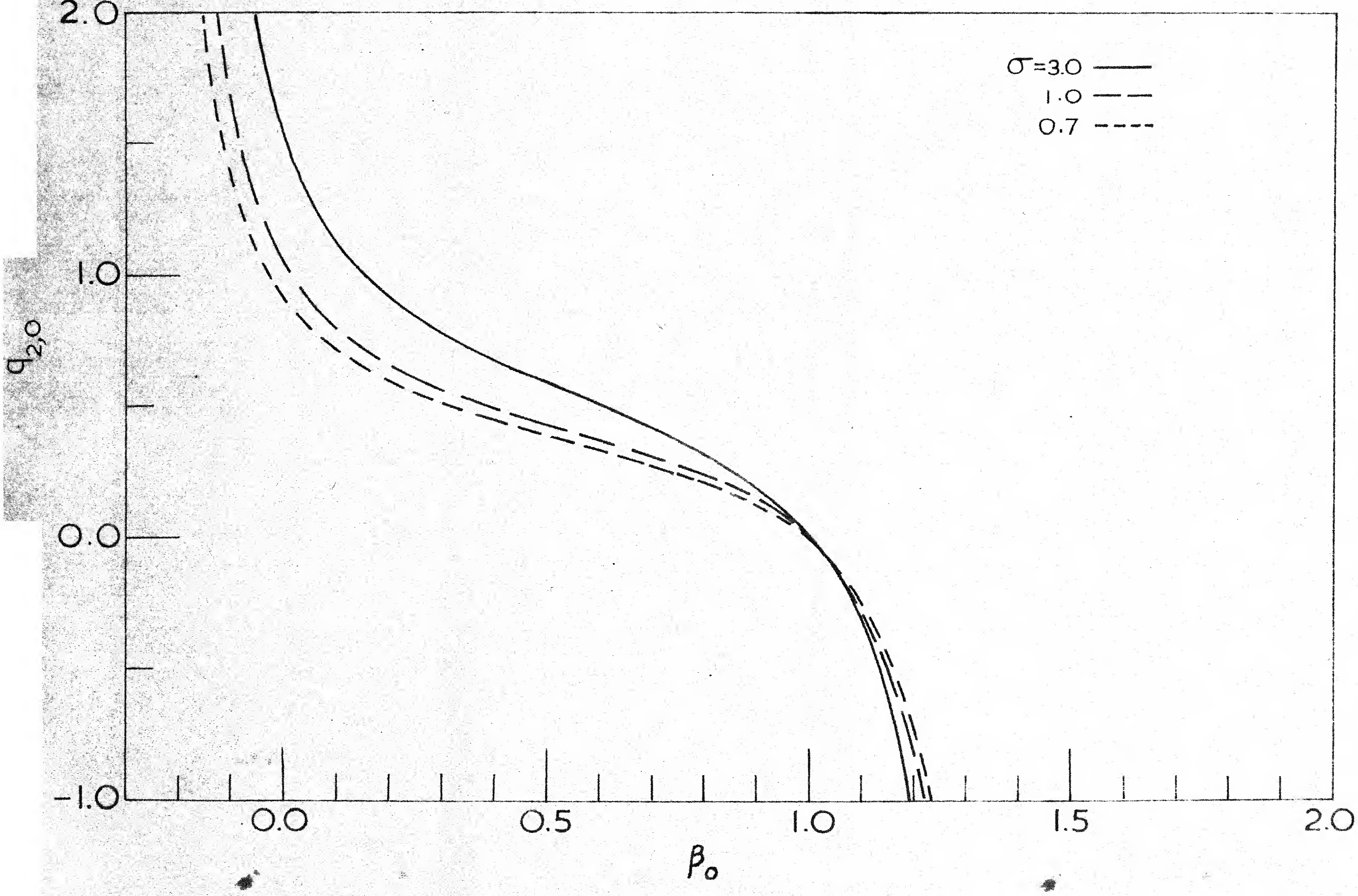


Fig. 3.30_Change in heat transfer due to external vorticity

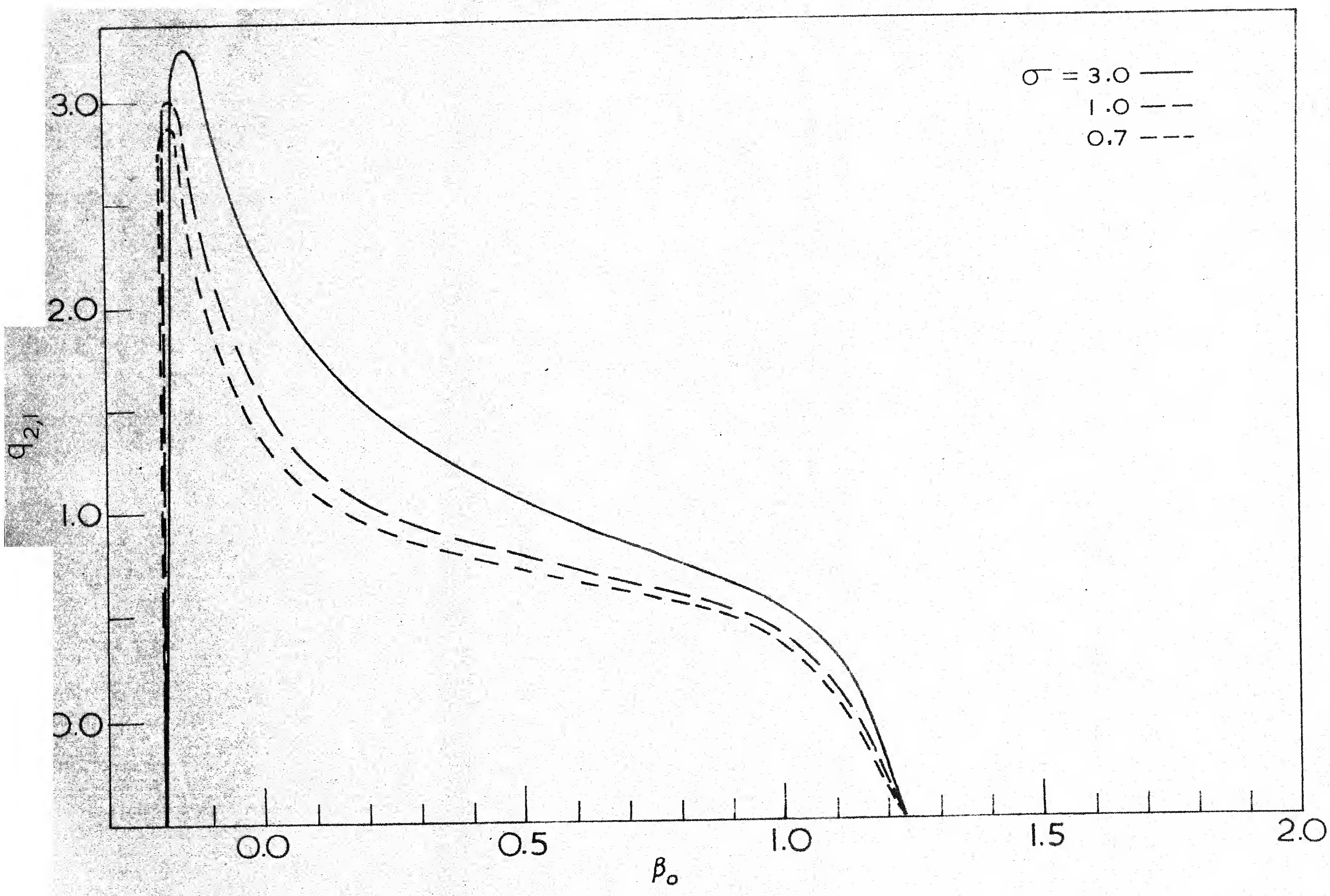


Fig.3.31_Change in heat transfer due to external vorticity

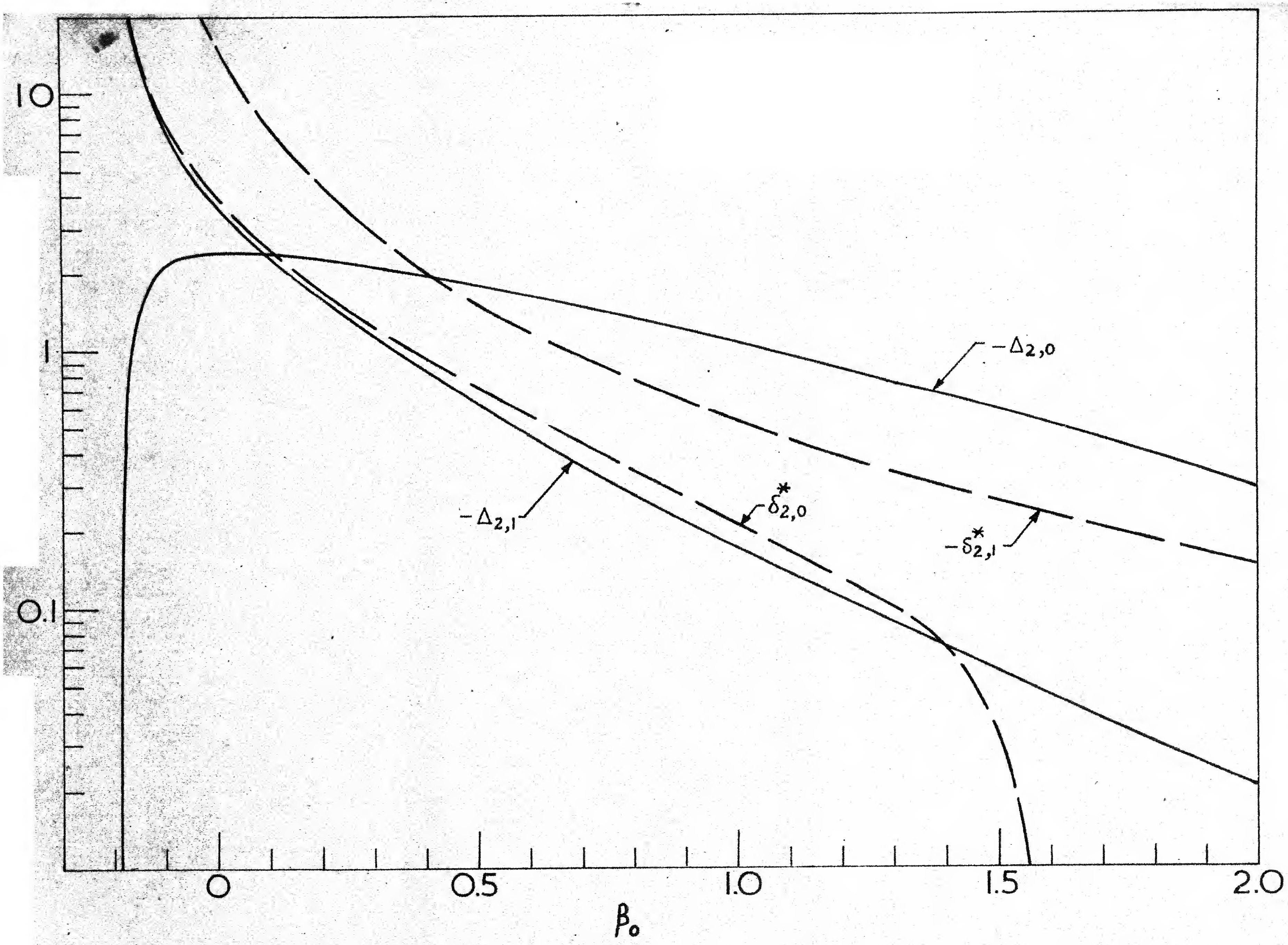


Fig. 5.32_Change in displacement & momentum thicknesses due to

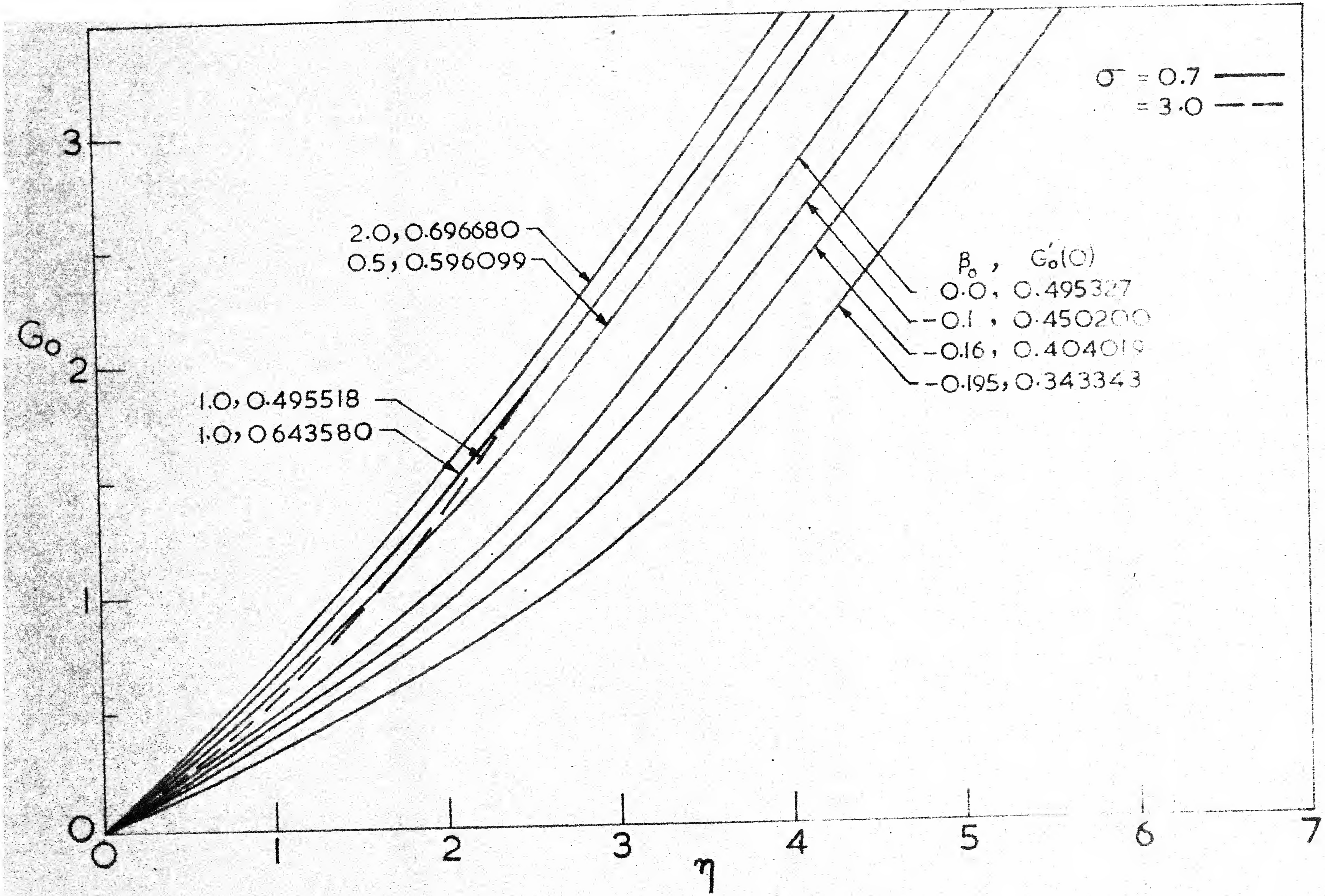


Fig.5.33_Change in temperature profiles due to stagnation enthalpy gradient (Eqn. 5.111)

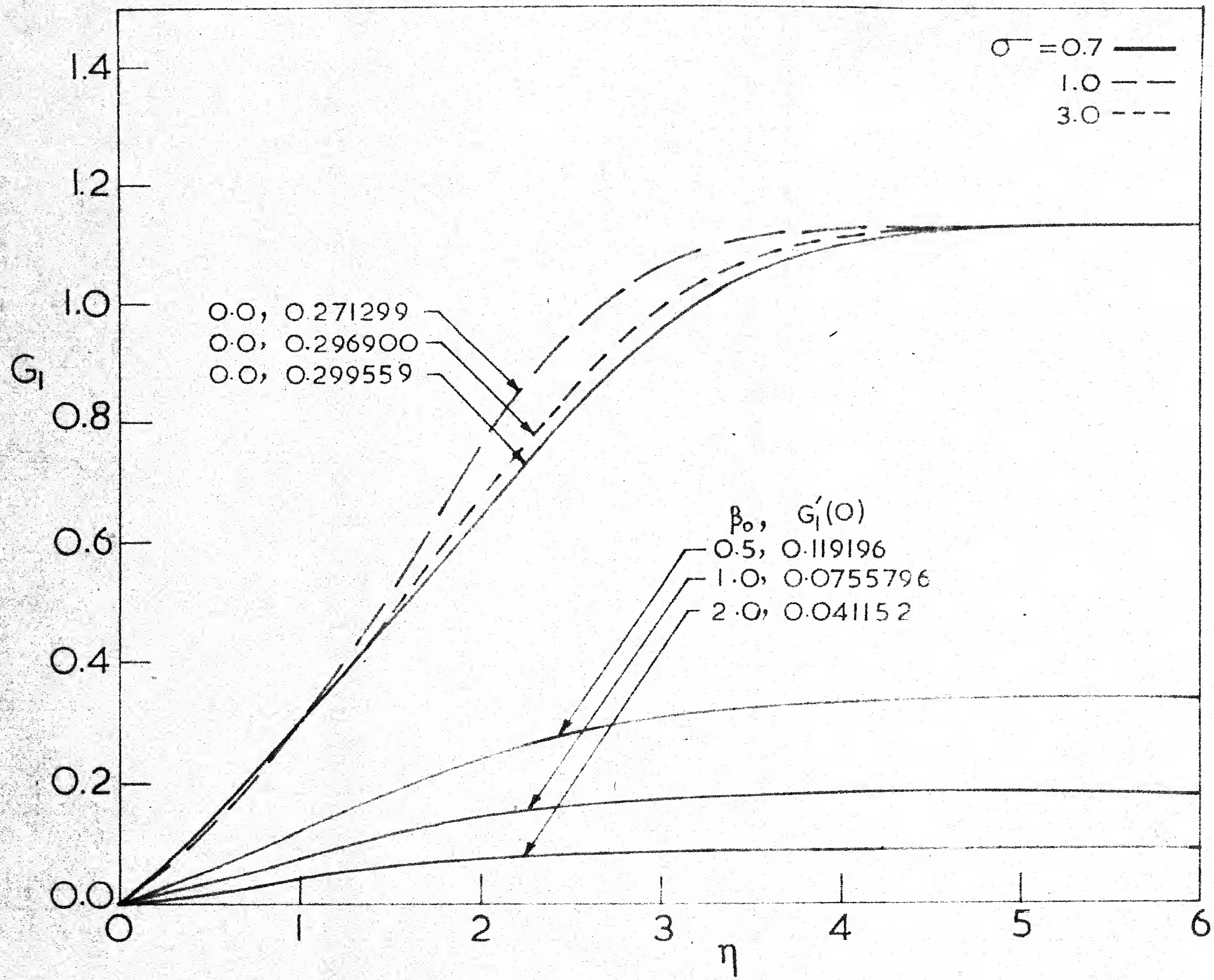


Fig 5.34_ Change in temperature profiles due to stagnation enthalpy gradients (Eqn. 5.112)

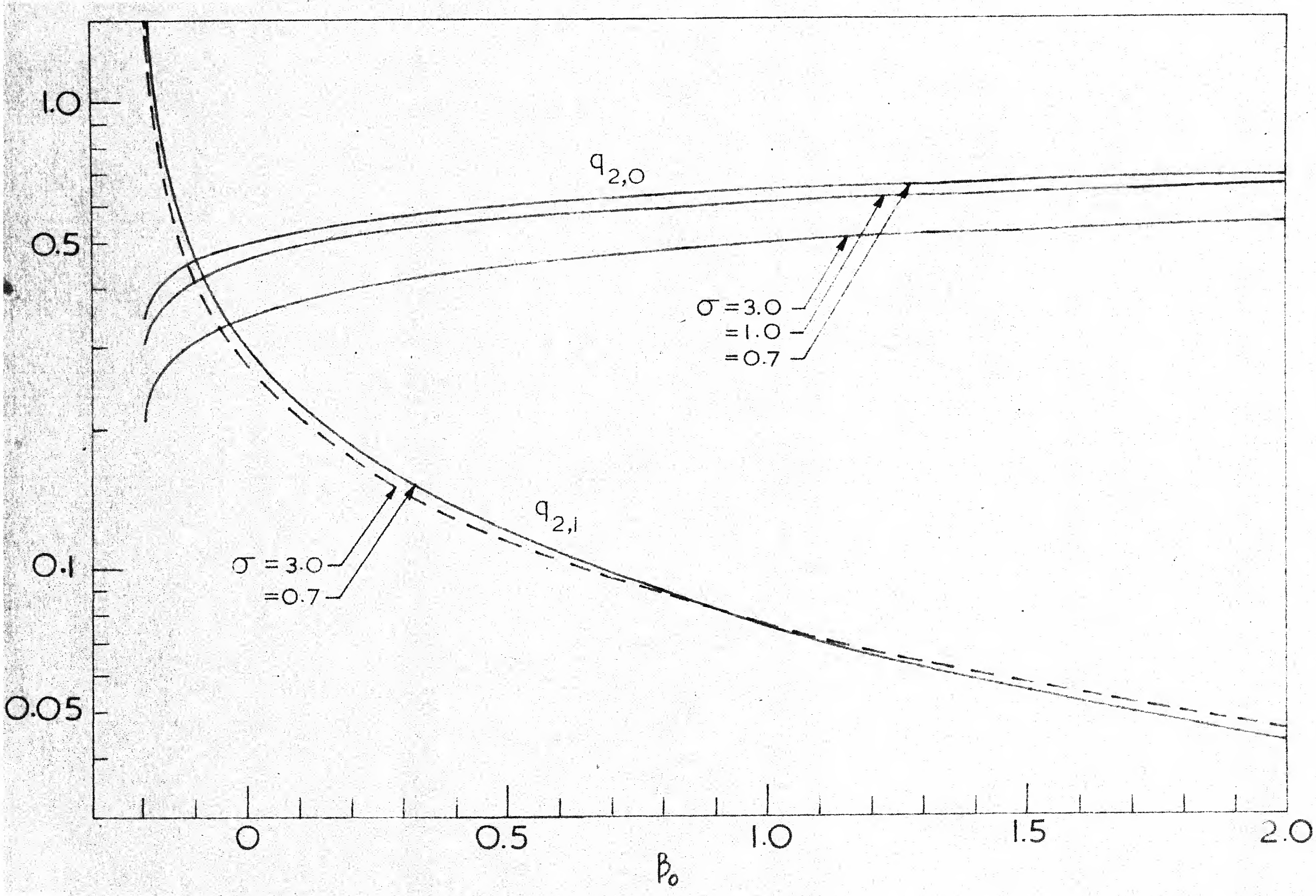


Fig.5.35_Change in wall heat transfer due to external stagnation

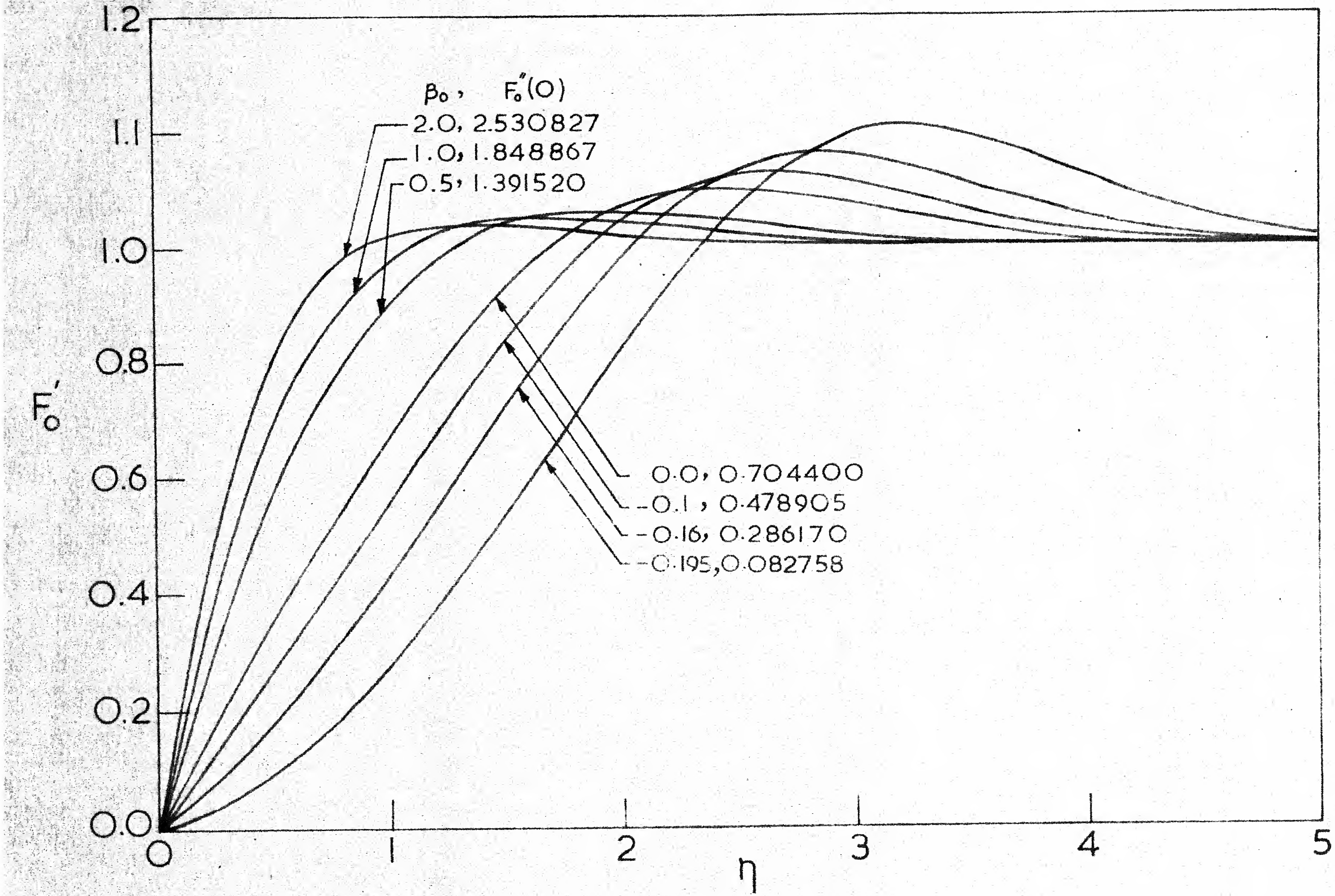


Fig. 5.36_ Change in velocity profile due to displacement speed (Eqn. 5.127a)

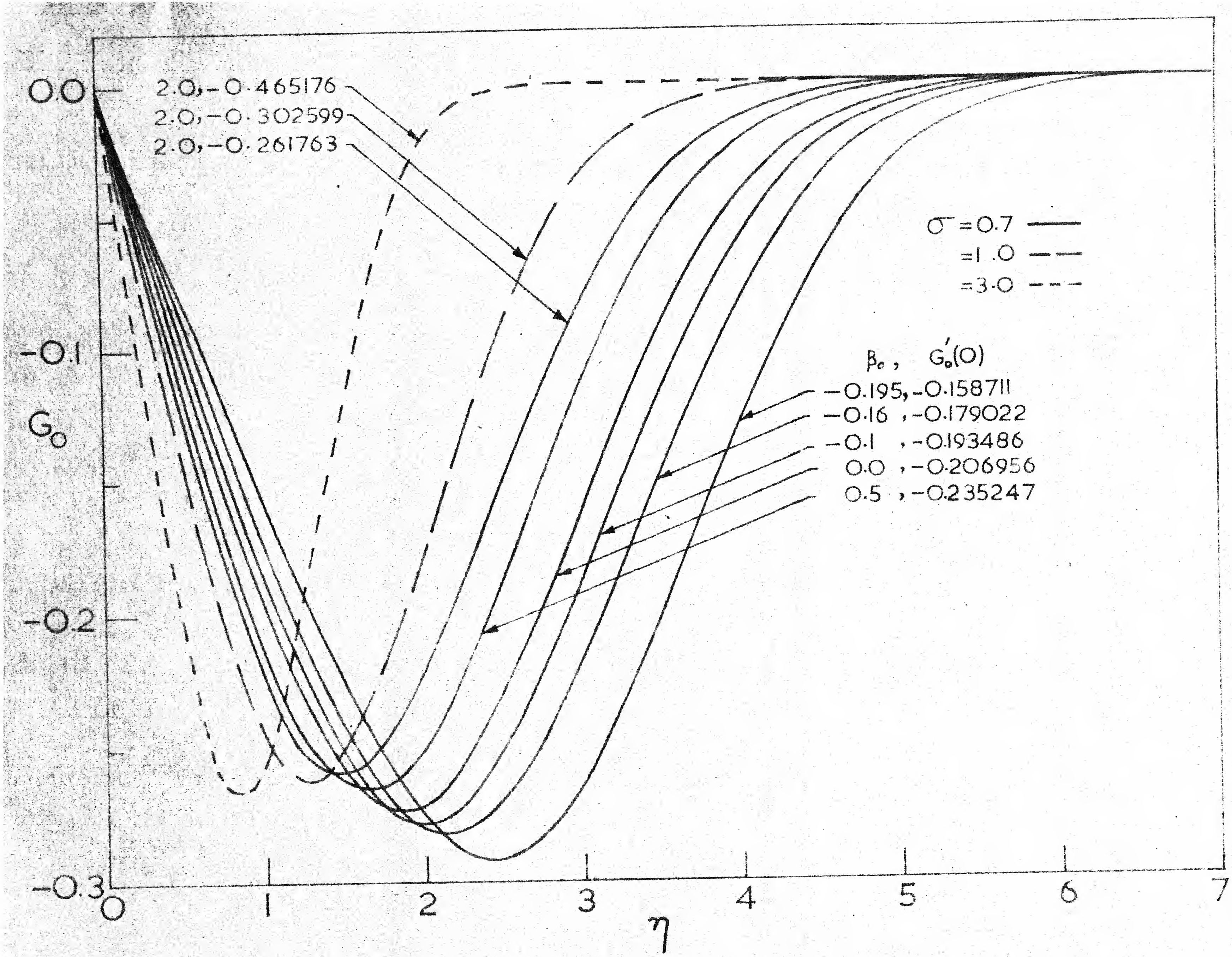


Fig.5.37_Change in temperature profiles due to

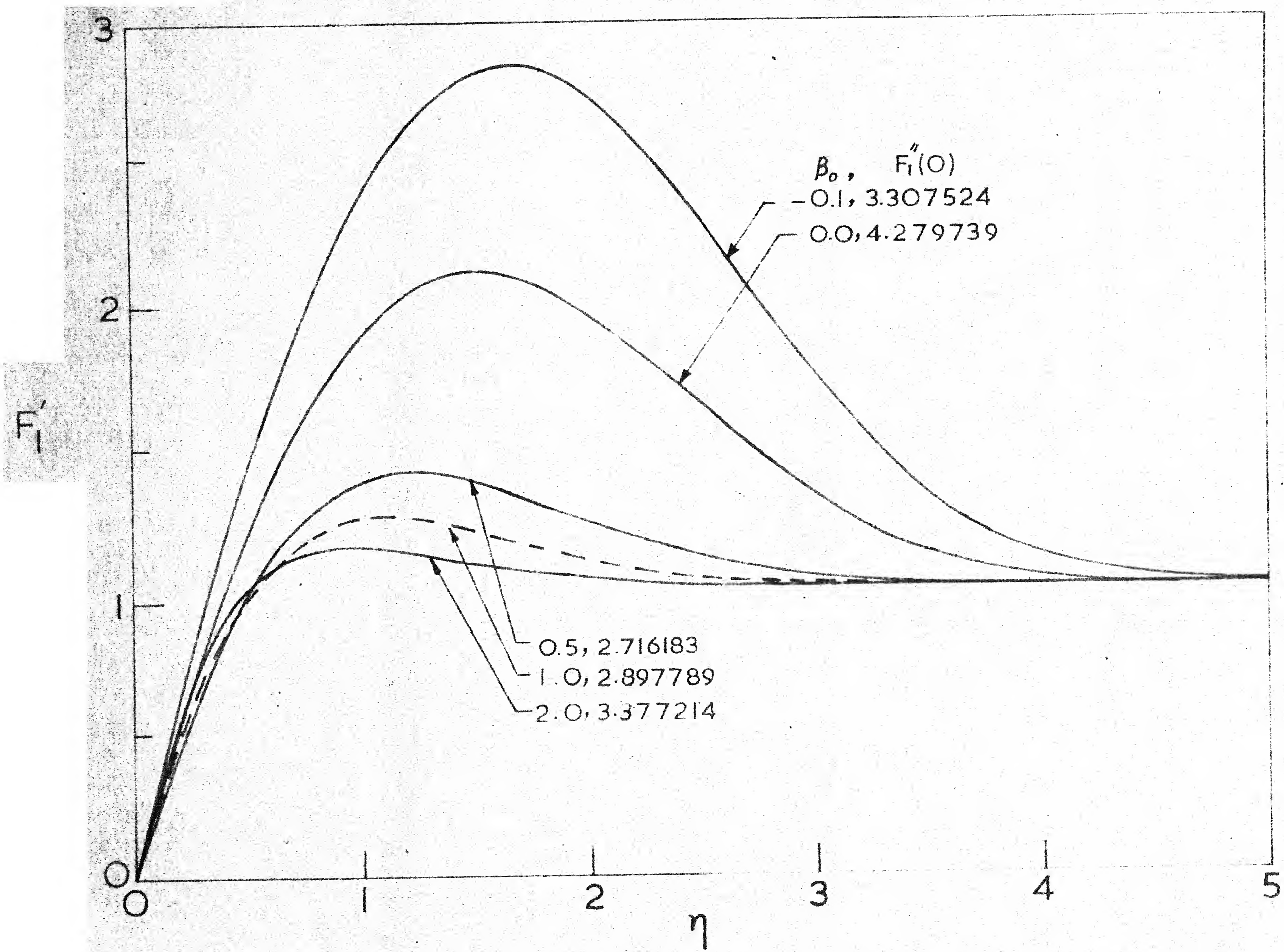


Fig.5.38_Change in velocity profiles due to displacement speed (Eqn.5.128a)

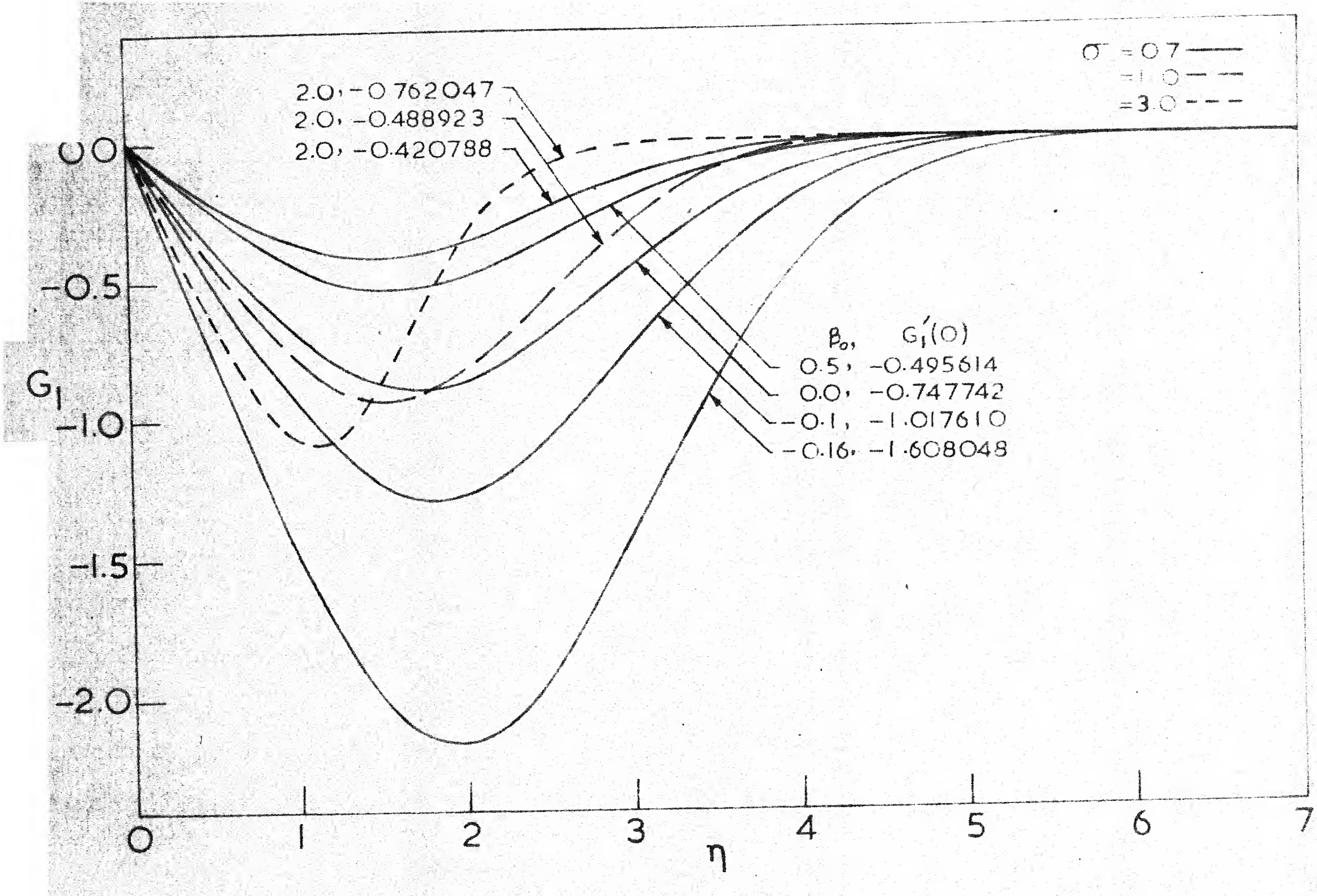


Fig. 5.39 Change in temperature profiles due to displacement speed (Eqn. 5.128b)

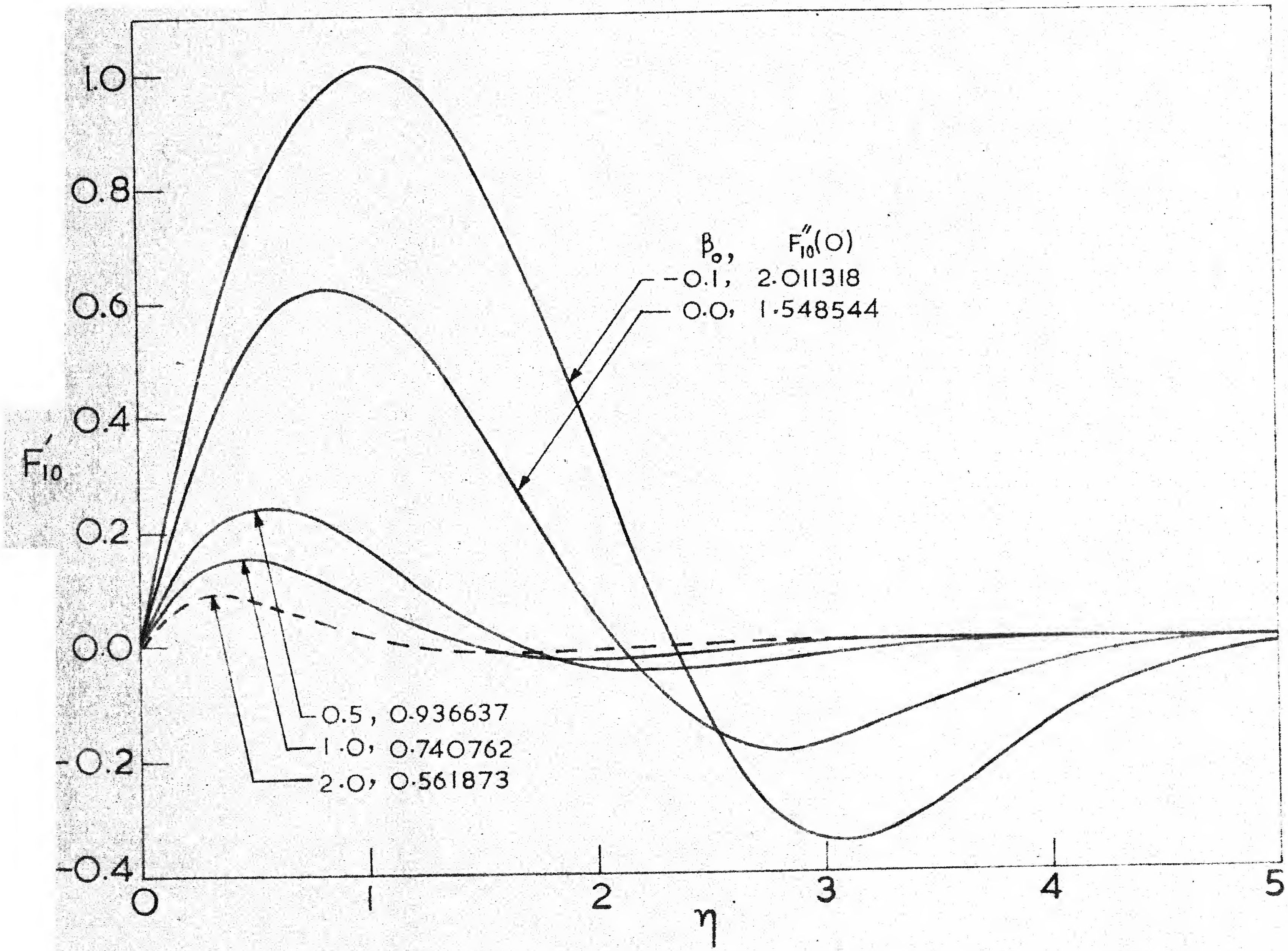


Fig.3.40 – Change in velocity profiles due to displacement

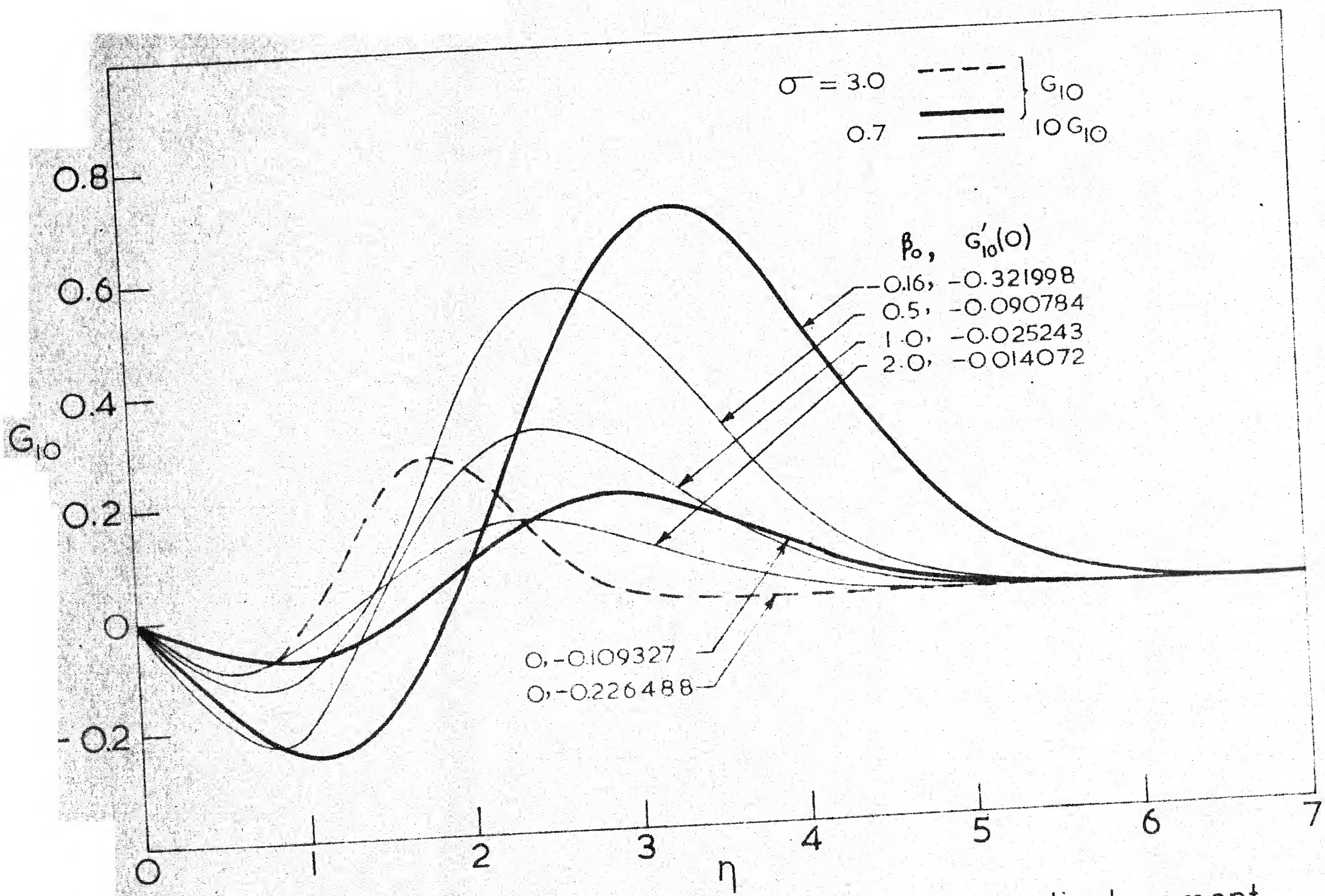


Fig.5.41 – Change in temperature profile due to displacement speed.

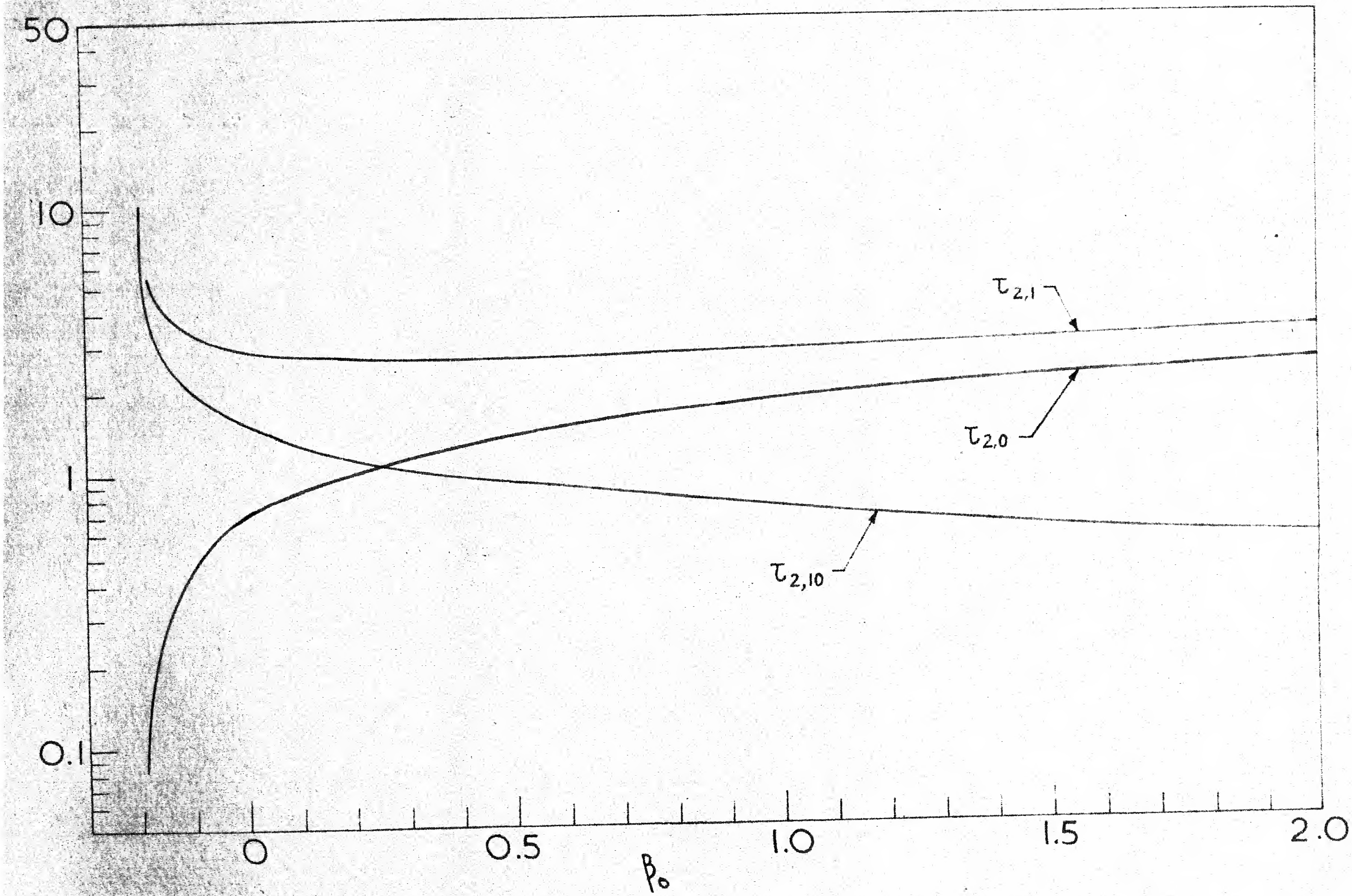


Fig 5.42 – Change in skin friction due to displacement speed.

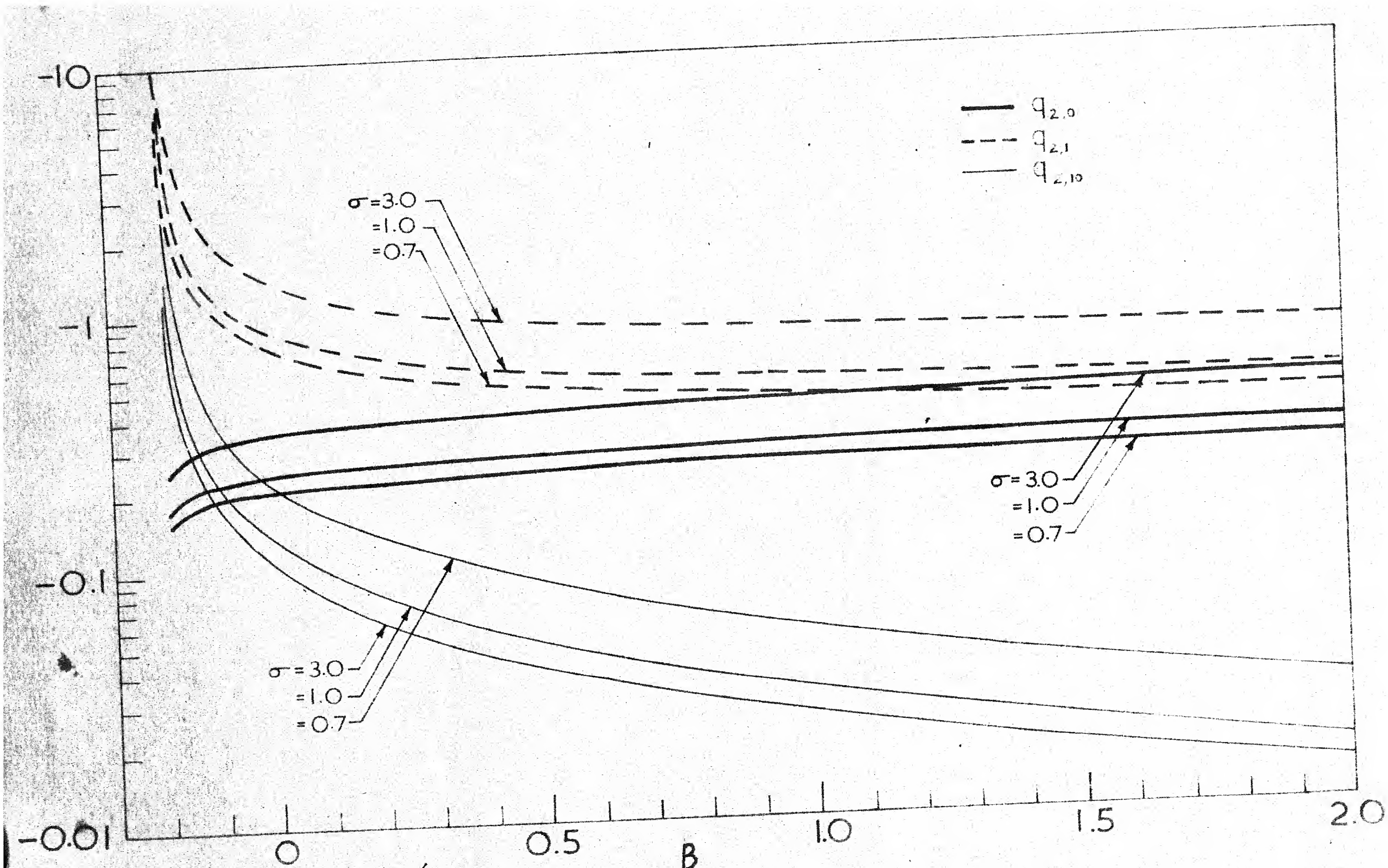


Fig.5.43_Change in wall heat transfer due to displacement speed.

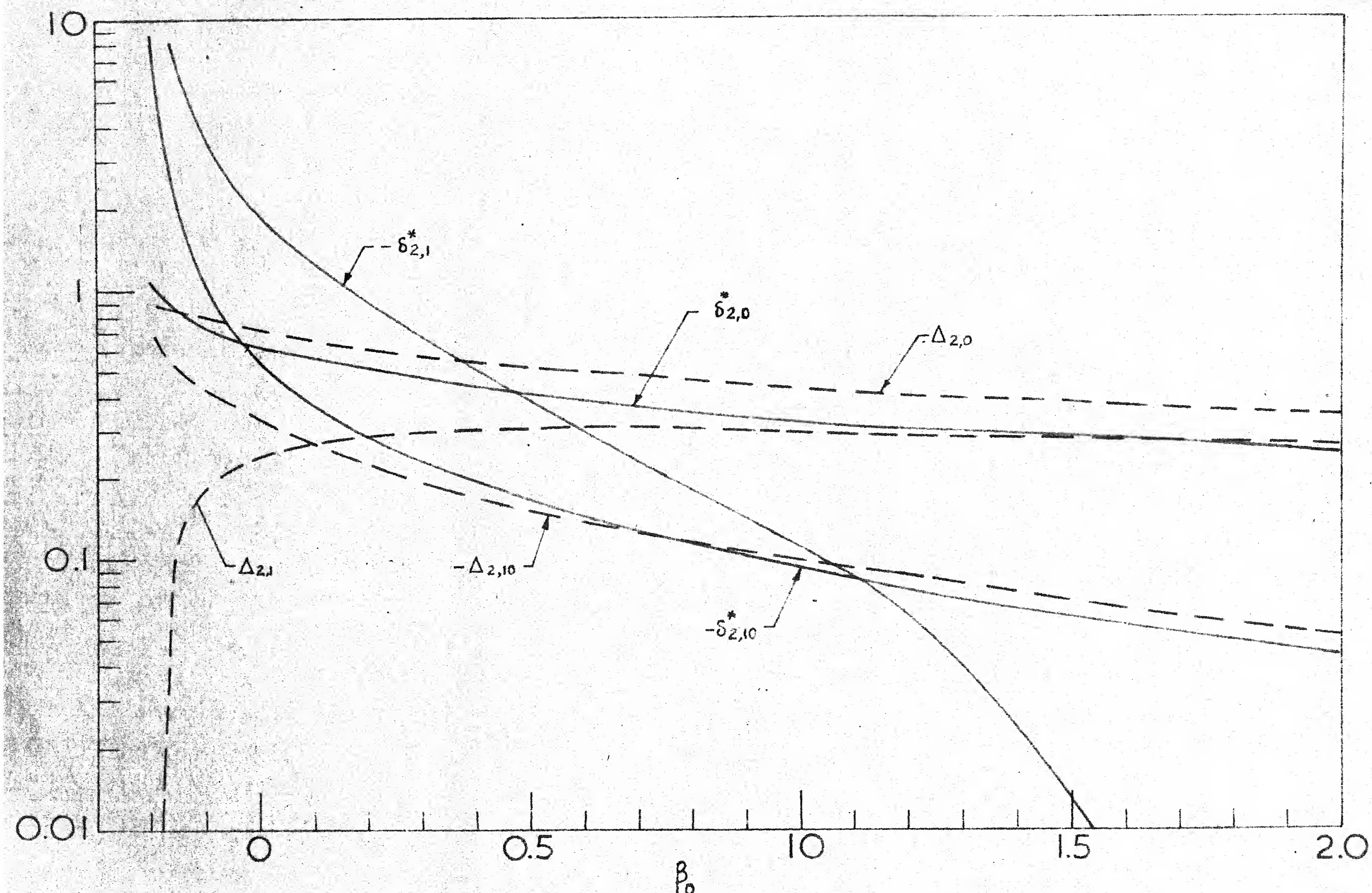


Fig.5.44_ Change in displacement and momentum thickness due to displacement speed.